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1801

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AN INTRODUCTION

TO

PHYSICAL MEASUREMENTS,

WITH APPENDICES ON

ABSOLUTE ELECTRICAL MEASUREMENT, ETC.

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NEW YORK:
D. APPLETON AND COMPANY,
549 & 551 BROADWAY.
1874.

PREFACE.

THE Author, in the preface to the second German edition, gives a sketch of the purposes which he hopes that the present book will serve. He says, a truth which all experience confirms, that the mere verbal teaching of physical laws is seldom of much use, tending frequently merely to confuse the student; while the simple performance of an experiment gives him confidence in himself and in the laws he is investigating. Another use of such a manual in the education of the scientific student is to lead him, by means of measurements which can be independently verified, to that knowledge of his powers which is so important when he has to do any original work. The greater part of the treatise is devoted to measurements of physical quantities. From this circumstance we have thought its object better expressed by the title we have placed at the head of it than by a literal translation of the German one.

Descriptions of apparatus are but rarely given, as students mostly have instruments provided for them, and seldom have to make their own apparatus, or to put it together.

The mathematical knowledge required is but very elementary, as the proofs of the formulæ are only given when they present no complex arguments.

The body of the work and the Appendix on Absolute Measure is, with little exception, an almost literal translation from the second German edition; but the Translators alone are responsible for the remaining Appendices and for several additional Tables.

THOMAS H. WALLER.
HENRY R. PROCTER.

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PHYSICAL MEASUREMENTS.

1.—Errors of Observation. Mean and Probable Error.

The numerical value of a physical quantity is affected with error from the inaccuracy of the observation. If the same quantity have been repeatedly measured, we require some means of calculating the most probable value in order to obtain an opinion as to the probable limits of error from the amount of agreement of the observations.

When all the separate determinations are, in the opinion of the observer, entitled to an equal degree of confidence, the arithmetical mean of the separate determinations gives, as is well known, the most probable value of the required quantity,—that is, all the separate values are added together, and the sum divided by the number of determinations.

We may here insist upon the fact that it is generally quite inadmissible arbitrarily to exclude from a series of observations some of the number, simply because they do not agree with the greater number. The probability of an increased error being introduced by the irregular numbers will be compensated by the very process of taking the arithmetical mean, for as single ones among a greater number they have a small influence upon the mean value.

If now the separate determinations be compared with the mean value, there will be found greater or less differences, from the amount of which the *probable error* of an observation as well as that of the result can be found by the following rules. First, the sum is taken of the *squares of the errors, i.e.* the difference between each separate observation and the mean is squared, and the resulting numbers added

together. The sum divided by the number of observations diminished by 1 gives the square of the mean error; the square root of it is the *mean error of a single observation.* If now this mean error be divided by the square root of the number of observations, we obtain what is called the *mean error of the result.*

Multiplying the mean error by 0·6745 (or $\frac{27}{40}$, or with sufficient accuracy for most purposes by $\frac{2}{3}$), we get the *probable error.* This last expression means that it is as likely that the actual unknown error is less than the "probable error" as it is that it is greater.

Let us then call

n the number of observations;
$\delta_1, \delta_2, \delta_3 \ldots \delta_n$ the deviations from the arithmetical mean;
S the sum of the squares of the errors;

$$i.e.—S = \delta_1^2 + \delta_2^2 + \delta_3^2 + \ldots . \delta_n^2;$$

then the mean error of a single observation $= \pm \sqrt{\frac{S}{n-1}}$; the mean error of the result obtained by taking the arithmetical mean

$$= \pm \sqrt{\frac{S}{n(n-1)}};$$

the probable error of a single observation

$$= \pm 0·6745 \cdot \sqrt{\frac{S}{n-1}};$$

the probable error of the result

$$= \pm 0·6745 \cdot \sqrt{\frac{S}{n(n-1)}}.$$

On the calculation of errors with several unknown quantities see (3).

It will be obvious that only that part of the error is expressed by quantities thus calculated, which is introduced by true uncertainty of observation—that is, by such errors as give too great a value as often as too small a value. But there may exist *constant errors,* the cause of which may be in the indications of the instrument, or which may be so related to them that the observer makes errors which

preponderate in one definite direction. It is an important problem, either to find out such errors and then correct the result, or to make such combinations of the results or such changes of method that the constant errors are thereby eliminated.

Example.—The density of a body was determined ten times, with the results given in the first column.

	Found.	Difference δ from the Mean.	δ^2.
	9·662	− 0·0019	0·000004
	9·673	+ 091	083
	9·664	+ 001	000
	9·659	− 049	024
	9·677	+ 131	172
	9·662	− 019	004
	9·663	− 009	001
	9·680	+ 161	259
	9·645	− 189	357
	9·654	− 0·0099	0·000098
Mean	9·6639		S = 0·001002

Then since $n = 10$ we have

the mean error of one observation $= \sqrt{\dfrac{0·001002}{9}} = \pm\, 0·011$,

" " the result $= \sqrt{\dfrac{0·001002}{10 \cdot 9}} = \pm\, 0·0033$,

probable error of one observation $= 0·6745 \sqrt{\dfrac{0·001002}{9}} = \pm 0·0071$,

" " the result $= 0·6745 \sqrt{\dfrac{0·001002}{10 \cdot 9}} = \pm 0·0023$.

According to this we may wager one to one that the error which affects the separate determinations of the density of this body, with the instruments, care, and experience supposed above, is less than 0·0071. It happens accidentally that just half the above differences are smaller, the other half greater, than this amount.

The probable error deduced from a series of only 10

observations can only be considered as an approximation. It was really superfluous to calculate it out to three places as we have done. Similarly the approximate value $\frac{2}{3}$ might have been used instead of 0·6745.

The determinations given above were made by different observers, using different sets of weights and different thermometers. Errors of the balance, which influence the determination of density in one direction only, are not taken into account. Constant errors would therefore be avoided in this example. But in order that the amounts of error calculated above should really represent the probable errors, we must be able to assume that all the observers took proper care in the removal of the bubbles of air which might have clung to the body when weighed in water. Otherwise the observations would be affected with an error, if not constant yet one-sided, for under the supposed circumstances the density must always come out too small. Errors of this sort cannot therefore show themselves in the differences from the mean value.

2.—Influence of Errors of Observation on the Result.

We frequently do not find a result directly by observation, but must deduce it from observed magnitudes, or even from several such, by calculation. Thus the density of a body is found from several weighings, the modulus of elasticity from measurings of length, the strength of a galvanic current from the deflection of a needle, according to certain formulæ. Hence arises the problem to determine to how great an extent the result will be in error when the observed magnitudes are affected by a certain error.

The object of this calculation of errors may sometimes even be to form a judgment as to the accuracy of the result. Further, we learn from it what abbreviation of the calculation we may allow ourselves without unduly increasing the inaccuracy. In cases where the measurement is the result of several observations, it also shows us over what part we

must expend the greatest care. Finally, it is frequently in our power to vary the proportions of the experiment in different ways: this calculation of errors alone gives us the information as to what choice of ratio is most advantageous, *i.e.* which gives the least influence to errors of observation upon the result.

Such are, for instance, the considerations from which the rule given on p. 138 is derived—that in determining the horizontal intensity of the earth's magnetism, it is best to take the distances of the deflecting magnet in the ratio 4 : 3. In the same way also are got the rules, that the measurement of the strength of a galvanic current with a tangent galvanometer furnishes the most accurate results with an angle of deflection of about 45°; that the two current strengths, from which the resistance (p. 162) or the electromotive force (p. 173) of a galvanic battery are determined are most advantageously in the ratio 1 : 2, etc.

If we call the observed magnitude x, the required result X, X will be some function of x—*i.e.* will be given by some mathematical expression in which x occurs. If now we call f the error of x, the error introduced by it into X which we call F, is found by putting $x + f$ instead of x in the expression from which X is calculated. We shall now have a result somewhat different from X, the correct value; the magnitude of this difference is manifestly the error F.

Since the errors of observation are small quantities, this calculation may be much simplified. We first note the following rules:—

1. In determining the errors in the result, it is sufficient to use an approximate value for the observed magnitude, which we have called x. Indeed we are always compelled to do so, since the true accurate value is not known.
2. Correction terms (4) which occur in the formula for the result X, may, if we are not inquiring into their influence, be neglected in calculating the error.
3. If a measurement depend on several independent observations, the final result will be an expression compounded of the separately observed quantities.

Several errors may be included in these. But if the influence of the errors introduced by one of the magnitudes is to be determined, the others need not be taken any account of.

4. The error in the result which arises from an error of observation varies proportionally with this latter. In other words, the difference which we have above, called F may be represented as a product of which the error f of the observed magnitude is one factor.
5. From this it follows also that the errors of the result which arise from errors of observation, equal in magnitude but opposite in sign, are also equal in magnitude but have contrary signs.

The calculation may almost always be made very much shorter by the use of approximation formulæ for calculating with small magnitudes. These may easily be constructed by the aid of the differential calculus. If f be the error which occurs in the observed value, the error F of the result X is obtained by multiplying the partial differential coefficient of X with regard to x by f. Therefore

$$F = f\frac{dX}{dx}.$$

In order to bring the expression for the error to a simple form, without the use of the differential calculus, it will, if not always yet very often, be possible to adopt the plan for the calculation of correction quantities given at the end of this article: by suitable transformations we must bring it to pass that the error of observation f occurs only as a small quantity added to or subtracted from 1, upon which for further reduction the formulæ given below, or special ones, may be at once used.

When the result has been got from several observations combined, we may, according to No. 3 (see p. 5), investigate the influence of the single errors separately. Each of them may of course make the result either too small or too great, and the total error will be larger or smaller according as the signs happen to be the same or different. The maxi-

mum of error will be obtained when the partial errors have the same sign. The error *probably* arising is found by adding the squares of the partial errors, and taking the square root of the sum. The employment of these rules in a special case will serve to explain this sufficiently.

We choose as our example the determination of the density of a solid body which sinks in water, by the ordinary method, in which the body is weighed in air and in water. We will determine the effect of an error in weighing upon the density deduced from this weighing. If we call the weight of the body in the air m, and the weight in water m', the density is

$$\frac{m}{m-m'}.$$

To this formula must of course be added the corrections depending upon the loss of weight in the air, and upon the expansion of the water; but, according to No. 2, p 5, we need not trouble with these in the simple calculation of the error.

According to No. 3 we may consider the errors in m and m' separately, since they are independent of one another. Let us therefore find first the influence upon the result of an error in the weight in air. If we had committed the error f in this weighing, we should, instead of the true weight m, have found $m+f$, and should therefore obtain the density $\frac{m+f}{m+f-m'}$.

Using formula 8, p. 11, we will write for this

$$\frac{m}{m-m'} \cdot \frac{1+\frac{f}{m}}{1+\frac{f}{m-m'}} = \frac{m}{m-m'}\left(1+\frac{f}{m}-\frac{f}{m-m'}\right) = \frac{m}{m-m'} - f\frac{m'}{(m-m')^2}$$

The first term of the last expression is, however, the true result; so that

$$F = -f\frac{m'}{(m-m')^2}$$

is the error produced by the error $+f$ in weighing the body in air. In other words: if, in determining the density of a body which weighs m in air and m' in water, the weight be observed too great by f, the result will, supposing everything else correct, be too small by $f\frac{m'}{(m-m')^2}$.

The differential calculus gives, without further trouble,

$$F = f\frac{d\frac{m}{m-m'}}{dm} = -f\frac{m'}{(m-m')^2}.$$

According to No. 5, p. 6, it is needless to investigate the effect of a weight found too small. If the error in weighing in the air be $-f$, the result would be too great by $f\dfrac{m'}{(m-m')^2}$.

Secondly, let us consider an error committed in the weighing in water, which we will call f'. Setting therefore $m'+f'$ instead of m', the result affected with the error will be, as above,

$$\frac{m}{m-(m'+f')} = \frac{m}{m-m'-f'} = \frac{m}{(m-m')\left(1-\frac{f'}{m-m'}\right)} =$$

$$\frac{m}{m-m'}\left(1+\frac{f'}{m-m'}\right) = \frac{m}{m-m'} + f'\frac{m}{(m-m')^2}.$$

That is to say, by observing the weight in water as too great by f', we shall make the result too great by $F' = f'\dfrac{m}{(m-m')^2}$.

If, finally, we inquire as to the total error, which is compounded of the two errors of observation f and f', this has obviously its maximum value $\pm\dfrac{m'f+mf'}{(m-m')^2}$ when either m was found too great and m' too small, or *vice versâ*. The *probable* total error is

$$\pm\sqrt{F^2+F'^2} = \pm\frac{\sqrt{(fm')^2+(f'm)^2}}{(m-m')^2}.$$

We will take in addition, as a numerical example, the determination of the density of the same body of which we have already spoken, p. 3. We have there determined the amount of the error by the difference of the results which we obtained from their mean value. We want now to see what amount of error is to be expected from inaccurate observation in the weighing.

The weight of the piece was, in round numbers,

In air = 243,600 mgrs.
In water = 218,400 mgrs.

The greatest error in weighing, with the balance made use of, with moderate care, for loads such as the above, may be reckoned at 5 mgrs. when weighing in the air, at 8 mgrs. when weighing in water; which latter operation, on account of the friction of the water, is less accurate, whence

$$f = 5 \text{ mgrs.} \quad f' = 8 \text{ mgrs.}$$

(The errors must be reckoned in the same units as the observed weights themselves.)

The stated quantities substituted in the formulæ given above give,

$$\text{as the error depending on } m, \pm \frac{5\cdot218400}{25200^2} = \pm\, 0\cdot0017 = F;$$

$$\text{,, \quad ,, \quad ,, \quad } m', \pm \frac{8\cdot243600}{25200^2} = \pm\, 0\cdot0031 = F'.$$

In the most unfavourable case the total error amounts to 0·0048, but in the most probable case $= \pm \sqrt{F^2 + F'^2} = \pm\, 0\cdot0035$.

If, therefore, single ones of the above given determinations give considerably greater differences, there must have been present other sources of error besides the uncertainty of the weighing—(bubbles of air, inaccuracy in determining the temperature, mistakes in reckoning up the weights).

As a second example, the measurement of the strength of a galvanic current i with the tangent compass may serve. If φ be the angle of deflection of the needle, we have

$$i = C \tan \varphi,$$

where C is a factor constant for the same instrument. If an error f occur in the reading off of the angle φ, the error F in i follows from

$$i + F = C \tan (\varphi + f),$$

or by formula 10 (below)

$$i + F = C \left(\tan \varphi + \frac{f}{\cos^2 \varphi}\right); \text{ therefore}$$

$$F = C \frac{f}{\cos^2 \varphi} = i \frac{f}{\sin \varphi \,.\, \cos \varphi} = i \frac{2f}{\sin 2\varphi}.$$

$\dfrac{2f}{\sin 2\varphi}$ is therefore the error, expressed as a fraction of i, which corresponds to an error f in the reading off. Hence we have the very important rule for the use of the tangent compass—that angles of about 45° are most advisable for the accuracy of the measurement. For the same error in reading off produces an error in the result dependent upon the deflection, being very large both for very small angles and for those of nearly 90°, and having a minimum value for $\varphi = 45°$.

Rules for Approximation in calculating with Small Quantities.

When, in a mathematical expression, some numbers occur which are always very small in comparison with others, and which therefore are reckoned as corrections, the expression may frequently be brought into a form more convenient for calculation by the use of formulæ of approximation. It will very frequently recommend itself as the simplest to first give the expression such a form that the corrections are contained in terms added to or subtracted from 1, and very small compared with 1; this is not unfrequently the form in which it is already given. It will then be frequently possible to make use of one of the following formulæ to simplify the expression.

In these formulæ let the magnitudes denoted by δ, ϵ, ζ ... be very small compared with 1, so small that their second and higher powers δ^2, ϵ^2 ... as well as their products $\delta\epsilon$, $\delta\zeta$ ... which, again, are very small compared with δ, ϵ ... themselves, may practically be completely neglected compared with 1.

If, for example, $\delta = 0{\cdot}001$, $\delta^2 = 0{\cdot}000001$; if further, $\epsilon = 0{\cdot}005$, $\delta\epsilon = 0{\cdot}00005$;—it often happens that things which affect a quantity to the extent of some thousandths are important, whilst some millionths more or less appear a matter of complete indifference. It is usually easy to measure a length of about 1 metre accurately to the tenth of a millimetre. It would not do, therefore, to neglect a correction of a thousandth of the length, or 1 mm. But one or several millionths of the total length—*i.e.* thousandths of a millimetre—will most rarely have any practical influence, since the errors of observation are much greater.

On this supposition it may be easily shown that the following formulæ hold good, in which the expressions to the right of the sign of equality will usually be more convenient for calculation.

Where the sign $\pm$ or $\mp$ is placed before a quantity, either the upper or lower sign must be taken all through the formula.

$$(1+\delta)^m = 1 + m\delta. \qquad (1-\delta)^m = 1 - m\delta. \qquad (1)$$

therefore in different cases

$$(1+\delta)^2 = 1 + 2\delta. \qquad (1-\delta)^2 = 1 - 2\delta. \qquad (2)$$

$$\sqrt{1+\delta} = 1 + \tfrac{1}{2}\delta. \qquad \sqrt{1-\delta} = 1 - \tfrac{1}{2}\delta. \qquad (3)$$

$$\frac{1}{1+\delta} = 1 - \delta. \qquad \frac{1}{1-\delta} = 1 + \delta. \qquad (4)$$

$$\frac{1}{(1+\delta)^2} = 1 - 2\delta. \qquad \frac{1}{(1-\delta)^2} = 1 + 2\delta. \tag{5}$$

$$\frac{1}{\sqrt{1+\delta}} = 1 - \tfrac{1}{2}\delta. \qquad \frac{1}{\sqrt{1-\delta}} = 1 + \tfrac{1}{2}\delta, \text{ etc.} \tag{6}$$

$$(1 \pm \delta)(1 \pm \epsilon)(1 \pm \zeta) \ldots = 1 \pm \delta \pm \epsilon \pm \zeta \ldots \tag{7}$$

$$\frac{(1 \pm \delta)(1 \pm \zeta)\ldots}{(1 \pm \epsilon)(1 \pm \eta)\ldots} = 1 \pm \delta \pm \zeta \mp \epsilon \mp \eta \ldots \tag{8}$$

Thus also we may, instead of the geometrical mean of two quantities only slightly different from each other, p_1 and p_2, use the arithmetical

$$\sqrt{p_1 \, p_2} = \frac{p_1 + p_2}{2} \tag{9}$$

Further, the trigonometrical formulæ of approximation are convenient—

$$\sin(x+\delta) = \sin x + \delta \cos x, \quad \cos(x+\delta) = \cos x - \delta \sin x, \quad \tan(x+\delta) = \tan x + \frac{\delta}{\cos^2 x}, \tag{10}$$

in which δ signifies a small angle measured in terms of the angle (57°·3), for which the arc is equal to the radius.

3.—Determination of Empirical Constants by the Method of Least Squares.

If the same magnitude have been measured several times, the arithmetical mean gives the most probable value. But frequently the required magnitude is not the immediate object of the measurement, but must be deduced, by calculation, from the observations, according to known physical laws, and then the arithmetical mean is not always sufficient to find the most probable result from repeated measurements.

Mathematically considered, the quantities sought occur here as constants in an equation which also contains the observed magnitudes. Not unfrequently other unknown constants occur in this equation, and are to be at the same time determined, or at least eliminated. For this purpose at least as many observations are required as there are unknown

quantities; and if there be only just as many, we must, by substituting the observed values in the mathematical expression, make as many equations as there are unknown quantities, and deduce the latter from them in the ordinary way. But when, for the sake of increasing the accuracy of the determination, a large number of observations has been made, we must, in order to utilise all the materials, employ another way—of which the work may be facilitated by various devices, especially by adapting the observations to a plan determined beforehand.

Nevertheless, these devices require very careful and circumspect consideration in order to exclude any uncertainty, and not unfrequently entirely fail us. Hence, it is important that the calculation of probabilities by the method of least squares should present a systematic course of proceeding by which the calculations may be made without any uncertainty. Of course it may frequently be found that by this method also we are led into tiresome calculations, and hence another proof of the advantage of a method which affords a plan completely thought out before the observations are made.

As an example we take the simple problem to determine the length of a rod at 0°, and its expansion for 1° of temperature, from a number of measurements at different temperatures. If we call the length at $0^\circ = a$, and the expansion for $1^\circ = b$, we have for the length y, at any temperature x

$$y = a + bx.$$

a and b are two unknown constants, for determining which two observations would be sufficient. Suppose, for example, we had observed the lengths y_1, y_2, at the temperatures x_1, x_2 respectively, we should have

$$y_1 = a + bx_1 \quad y_2 = a + bx_2,$$

therefore $a = \dfrac{x_1 y_2 - x_2 y_1}{x_1 - x_2}$, $b = \dfrac{y_1 - y_2}{x_1 - x_2}$.

But more than two observations may have been made; suppose besides the pairs given above, x_3, y_3, x_4, y_4, etc. If the observations were free from error, the quantities sought, a and b, would have

the same numerical value when calculated from any two pairs; and, on the other hand, every value of y, calculated by this formula from the corresponding value of x, would be identical with the observed value. But, in reality, we find that on account of errors no values for a and b completely satisfy *all* the observations.

The fundamental law of the method of least squares is: The constants must be so chosen that the sum of the errors is a minimum. That is to say, with every different value of the constants the values calculated from the law by means of them will differ from the observed values by different amounts (the errors). The most probable values of the constants are found when the sum of the second powers of all the differences is the smallest possible number.

If we denote the mathematical expression of known form, which gives the dependence of the observed magnitude y, on another, x (or on several others), by the general expression $f(x)$, the magnitudes we seek occur in it as constants which we call $a, b, \ldots$ Our equation then is

$$y = f(x).$$

Let several values $y_1\ y_2\ y_3 \ldots$ be observed corresponding to the known values $x_1\ x_2\ x_3 \ldots$ By the above law the numerical values of $a, b \ldots$ are to be so determined that when they are substituted in $f(x)$, the sum of the squares of the differences between the calculated and observed values has the smallest value possible. Therefore we must have

$$\{y_1 - f(x_1)\}^2 + \{y_2 - f(x_2)\}^2 + \{y_3 - f(x_3)\}^2 + \ldots + \{y_n - f(x_n)\}^2 = \text{a minimum.}$$

or, introducing the symbol of summation,

$$\Sigma\{y - f(x)\}^2 = \text{a minimum.}$$

We must keep in mind that all the values of x and y are known observed quantities.

By a law of the differential calculus, this condition produces as many equations as there are quantities $a\ b \ldots$ to be determined. We differentiate the expression $\Sigma\{y - f(x)\}^2$ with respect to $a, b \ldots$ considering these as the variables, and equate each partial differential coefficient to zero.

The equations from which $a, b \ldots$ are to be determined become therefore

$$\frac{d\,\Sigma\,\{y-f(x)\}^2}{da}=0 \quad \frac{d\,\Sigma\,\{y-f(x)\}^2}{db}=0, \text{ and so on.}$$

We have thus found a way, free from any uncertainty, by which we can make equal use of as many observations as we please.

Of course it may happen, with complicated forms of $f(x)$, that the equations derived by differentiation with respect to $a, b \ldots$ are not capable of direct solution. In such cases we must find a solution by trial and approximation. In the important case, however, where $f(x)$ has the form, $f(x) = a + bx + cx^2 + dx^3 + \ldots$ the direct solution is always possible.

Let us illustrate the problem by the example given above. Let the lengths of the rod observed at $x_1, x_2, x_3 \ldots x_n$, be $y_1\ y_2\ y_3 \ldots y_n$. According to the law of expansion with temperature $y = a + bx$, and so what we have above called $f(x)$ is here $f(x) = a + bx$. We have therefore to determine a and b, so that $(y_1 - a - bx_1)^2 + (y_2 - a - bx_2)^2 + \ldots + (y_n - a - bx_n)^2 =$ a minimum, or briefly $\Sigma\,(y - a - bx)^2 =$ a minimum.

Differentiation gives

$$\begin{array}{ll} \text{with respect to } a & \Sigma\,(y-a-bx)=0 \\ \text{with respect to } b & \Sigma x\,(y-a-bx)=0 \end{array}$$

or, observing that with n observations $\Sigma a = an$,

$$\Sigma y - an - b\,\Sigma x = 0$$
$$\Sigma xy - a\,\Sigma x - b\,\Sigma x^2 = 0.$$

By solving these equations with respect to a and b, we have

$$a = \frac{\Sigma x\,\Sigma xy - \Sigma y\,\Sigma x^2}{(\Sigma x)^2 - n\,\Sigma x^2}$$

$$b = \frac{\Sigma x\,\Sigma y - n\,\Sigma xy}{(\Sigma x)^2 - n\,\Sigma x^2}.$$

As an example, suppose the length of a measuring rod, which is to be corrected by comparison with a normal scale (the readings

of which have been already reduced by its known coefficient of expansion to its normal temperature), has been found

at temperature $x =$	20°	40°	50°	60°
the length	$= 1000{\cdot}22$	$1000{\cdot}65$	$1000{\cdot}90$	$1001{\cdot}05$ mm.

In order to shorten the calculations we take as y only the observed excesses of the length above 1000 mm. We shall then have for a the excess of the length at 0° above 1 metre.

The calculation is performed as follows:—

x	y	x^2	xy
20	$+ 0{\cdot}22$	400	$4{\cdot}4$
40	$0{\cdot}65$	1600	$26{\cdot}0$
50	$0{\cdot}90$	2500	$45{\cdot}0$
60	$1{\cdot}05$	3600	$63{\cdot}0$
$\Sigma x = 170$	$\Sigma y = 2{\cdot}82$	$\Sigma x^2 = 8100$	$\Sigma xy = 138{\cdot}4$

$$\text{therefore } a = \frac{170 \,.\, 138{\cdot}4 - 2{\cdot}82 \,.\, 8100}{170^2 - 4.8100} = -\,0{\cdot}196 \text{ mm.}$$

$$b = \frac{170 \,.\, 2{\cdot}82 - 4 \,.\, 138{\cdot}4}{170^2 - 4 \,.\, 8100} = +\,0{\cdot}212 \text{ mm.}$$

The length of the rod at 0° is therefore $999{\cdot}804$ mm., and at the temperature t, $999{\cdot}804 + 0{\cdot}212\,t$.

If now the lengths are calculated for 20°, 40°, 50°, 60°, we shall find—

x	y Calculated mm.	Observed mm.	Error Δ mm.	Δ^2
20°	$1000{\cdot}228$	$1000{\cdot}22$	$+ 0{\cdot}008$	$0{\cdot}000064$
40	$1000{\cdot}652$	$0{\cdot}65$	$+ 0{\cdot}002$	0004
50	$1000{\cdot}864$	$0{\cdot}90$	$- 0{\cdot}036$	1296
60	$1001{\cdot}076$	$1{\cdot}05$	$- 0{\cdot}026$	0676
				$\Sigma\Delta^2 = 0{\cdot}002040$

The student may verify that any alteration of a or of b increases the sum of the squares of the errors.

Exactly the same method of proceeding would be employed to find the modulus of elasticity from a number of observations on the length of a rod when stretched by different weights, or to determine the relative rate of two clocks from several comparisons between them; in fact, wherever two quantities increase proportionally with one another.

The expansion of most fluids by heat is irregular; the natural law is, however, not known. In this case, and in many similar ones, we usually make use of an algebraical expression of a higher degree as an approximation—*e.g.*, $y = a + bx + cx^2$. The determination of a, b, c from any number of observations is precisely the same as that given above.

The so-called *mean error of an observation* is obtained in this problem from the sum of the differences between observed and calculated magnitudes, if n = number of observations, m that of the constants $a, b, c \ldots$ to be determined, as

$$\pm \frac{\sqrt{\Delta_1^2 + \Delta_2^2 + \ldots \Delta_n^2}}{n - m}.$$

Therefore in the above example, where $n = 4$, $m = 2$, we have

$$\pm \sqrt{\frac{0 \cdot 00204}{4 - 2}} = \pm 0.032 \text{ mm.}$$

4.—Corrections and the Calculation of Corrections.

In almost the whole range of practical physics corrections come in as a general, very inconvenient element. From their importance these corrections require special mention. The result sought is almost never given directly from the observations; much more frequently these are affected by circumstances which must not be neglected in accurate determinations. With greater pretensions to accuracy, the number of influencing circumstances which must be considered increases as well as the difficulty of eliminating them, so that frequently the most important part of the work is introduced by these *corrections*. Hence also it is necessary to be able easily to make allowance for such corrections, and to take them into the calculation, so far as is necessary, in as simple a manner as possible. How far we may go in taking account of corrections depends of course upon the limit which is here also imposed upon us by the deficiencies of the observations, as well as by our incomplete knowledge of the

laws of nature and of the numerical values which they involve. But, on the other hand, it is frequently unnecessary to carry the accuracy of the correction to this limit. It is very much oftener sufficient to attain to such a degree of accuracy that the neglected part of the corrections is materially less than the possible influence of the errors of observation upon the result. Hence come certain rules for the calculation of corrections, by which the processes may be much shortened and simplified without the result suffering any injury. Practice in these frequently-occurring calculations is an essential condition for accurate and yet ready physical work.

One of the simplest physical measurements, for example, is weighing or determining the mass of a body. If we take this directly from the observations, we have first the errors of observation, which are made up of those due to the inaccuracy of our readings, and of our judgment about them, and to some faults, not to be calculated on, of the balance—as friction, change of the ratio of the balance-arms, etc. It is also impossible to get or to make a set of weights free from error. And as we do not suppose specially good instruments or accurate observations, other errors, unavoidable, but determinable in their amount, and therefore to be eliminated from the result, become noticeable in the data given directly by the balance. It is therefore always requisite to take account of them, where we make any pretensions to accuracy. To this part belongs the inequality of the arms of the balance, which, at least with large weights, has usually a marked influence. It is eliminated by the rules given in (9) and (10) already abbreviated for use.

But, secondly, the weights and the body weighed suffer a loss of weight on account of the air which they displace, which—even in the use of ordinary shop-scales, which show 1 grm. with loads of 1 kgr.—may become greater than the errors of weighing. In order, now, to apply this correction (to reduce to the weight *in vacuo*), we must know the density of the air, a magnitude which may vary within certain limits. But although the complete neglect of the correction is only

admissible in a very rough weighing, it is, on the other hand, easily seen that for common use, even in scientific investigations, the alteration of the density of the air is not of sufficient importance to be considered, and we may give a mean value to the correction. If, therefore, we confine ourselves to a correspondingly approximate calculation of the correction, a very considerable improvement of the result may be effected in about a minute.

Of course, the labour is greater, if the mean value will not be sufficient. In this case, the temperature and height of the barometer, at least, must be observed, from which the density of the air may be found in Table 6. With a further increase of pretension to accuracy, the observed height of the barometer must not be taken as the real height, since mercury expands by heat; this expansion must be taken into consideration [to reduce the barometer reading to 0° (20)]. The same is true of the scale on which the height is measured. The variation of gravity on the earth's surface would also have to be brought into the calculation. Finally, the density of the air varies with the constantly-present but variable amount of vapour of water, and therefore in very accurate weighings this also must be taken into the reckoning.

Now, if all these observations and calculations were carried out with complete accuracy, they would become very laborious. But now, what we have said above comes into use. After we have informed ourselves as to what degree of accuracy we desire or can attain in the result, and as to the influence of the corrections, we find that an approximation is always admissible with this latter, and, with some practice, succeeds in its object with small trouble.

In the same way, corrections come into most physical problems. It is especially the changing *temperature* which, in many ways, influences the measurements, and therefore frequently furnishes a reason for corrections.

It is usually possible to make use of the processes described on p. 10, and the formulæ for approximation there given, for shortening the calculation of corrections. Almost every physical problem furnishes examples for practice.

Examples.

(1.) It is well known that we call 3α the coefficient of the cubical expansion of a body, when α is used for the linear coefficient. Strictly speaking, when the linear dimensions are varied in the ratio $1 + \alpha t$, the volume changes in the ratio $(1 + \alpha t)^3 = 1 + 3\alpha t + 3\alpha^2 t^2 + \alpha^3 t^3$. But for all solid bodies $\alpha < 0{\cdot}00003$, so that even for a change of temperature of $100°$, the neglected part $3\alpha^2 t^2 < 0{\cdot}000027$, or $\frac{1}{37000}$ of the total. Therefore it is only when such small quantities are under consideration that the abbreviated calculation must not be employed. Then, however, it must also be taken into the calculation that the coefficient of expansion itself varies a little with the temperature. The term $\alpha^3 t^3$ is entirely without noticeable influence.

(2.) In (20) we treat the expansion of the mercury as a correction by putting $\frac{l}{1 + 0{\cdot}00018\,t} = l - 0{\cdot}00018lt$ (formula 8, p. 10), in the reduction of the height of the barometer to $0°$. Here we neglect the higher powers of $0{\cdot}00018\,t$. But it will be seen that the next power amounts for $t^\circ = 30$ to only $0{\cdot}00003$; therefore multiplied by $l = 760$ mm., about $\frac{1}{45}$ mm., a quantity which may almost always be neglected.

On the other hand, it would not be allowable to treat the expansion of a gas, which is about twenty times greater, as a correction.

(3.) When the weight of a body has been determined by double weighing, in order to eliminate the inequality of the arms, and the weights have been found on the one side p_1, on the other p_2, the actual weight is, strictly speaking, $\sqrt{p_1 p_2}$. Instead of this geometrical mean, the arithmetical $\frac{1}{2}(p_1 + p_2)$ may without hesitation be used (formula 9, p. 11). For, calling $p_1 = p + \delta$, and $p_2 = p - \delta$, for which $p = \frac{1}{2}(p_1 + p_2)$, we have

$$\sqrt{p_1 p_2} = \sqrt{p^2 - \delta^2} = p\sqrt{1 - \frac{\delta^2}{p^2}} = p\left(1 - \frac{1}{2}\frac{\delta^2}{p^2}\right). \quad \text{(Formula 3.)}$$

Now the balance must be very badly adjusted for δ to be as much as $\frac{1}{1000}p$. In this case $\frac{1}{2}\frac{\delta^2}{p^2}$ would be half a millionth—a quantity which, in comparison with 1, need never be considered if such a balance be used.

Other examples will be found below in the different problems.

5.—Rules for the Numerical Calculations.

The numerical calculation of the result can never be performed but with a limited number of figures, a circumstance which renders absolute accuracy in the calculations impossible. Here also it is important to obtain the required accuracy without unnecessary trouble.

It is generally well to keep to the rule that the result is to be brought out to so many figures; that the last of them, on account of the errors of observations, makes no pretension to accuracy, but that the last but one may be taken as pretty accurate. In doubtful cases one place too many should be taken rather than one too few.

All the figures, however, should be correct as to the working. Hence it follows that the calculation must be gone through with at least one place more than is to be given in the result, for by the neglecting of further figures the last place may by degrees become wrong to the amount of several units. The extra place is discarded in the result at the end, increasing the last but one by 1 if the figure rejected be 5 or more.

Of course ciphers added, or those prefixed to a decimal, will not be counted in the number of figures.

Example.—The determination of the density of the body already mentioned so many times (p. 3) gives the second place pretty accurately, but the third not so much so. This, therefore, is as far as we must go. In the mean value from ten observations, however, one more place may be given. We use here 5-figure logarithms for calculating the result, since 4 figures are to be correct.

6.—Adjusting and Testing a Balance.

The descriptions which follow refer, so far as any special construction is kept in view, to the form of the balance commonly used in chemical analysis.

Adjustment of the Balance.—There is usually a level or plumb-line attached to the balance-stand by the maker, by

means of which the adjustment is made with the foot-screws. Where this arrangement is wanting a level is placed in the balance-case, and the adjustment attained by it.

The beam is now released, and correcting any slight excess of weight on either side by changing the position of the sliding weight provided for the purpose, or, by adding small weights, the observer makes sure that the balance has a stable position of equilibrium. Should the equilibrium be unstable (the balance "*set*"), the movable weight in the middle must be screwed down until the defect is remedied.

The sensitiveness (amount of deflection for 1 mgr.) can be regulated by screwing up the movable weight as far as is necessary. The time of an oscillation increases with the increase of sensitiveness; it should be made, in balances of the ordinary form, about 10 to 15 seconds. Slower oscillations occasion loss of time in weighing, and usually cause irregularities in the adjustment which make the larger deflection useless.

When a suitable time of oscillation has been attained, the pointer is made to point to the middle division of the divided scale, or swing equally on both sides of it, when the balance is unloaded. This is effected by altering the place of the weight which is movable along the beam. We need not be afraid, when the desired object is nearly attained by means of the movable weight, of facilitating the last fine adjustment to the zero by performing it with the foot-screws, shortening one by as nearly as possible the same amount that we lengthen the other.

Testing the Balance.—The balance must fulfil the following conditions before it is taken into use:—

It must, when repeatedly stopped and again released, always take up the same position (it must be seen that the three knife-edges are carefully cleaned).

When the balance is swinging freely, the distance it swings should diminish but slowly.

When the stopping apparatus is raised, the pointer should stand immediately over the middle division; and when

it is lowered, the two pins on which the beam rests when stopped should release it at the same time.

The above conditions must still be fulfilled when the balance is loaded with the greatest weight for which it is safe to use it. We must in this case test with especial care the stability of the equilibrium, the constancy of the zero, and the slow diminution of the swing.

The equality of the arms should then be made sure of, which is known to be the case when weights (which should not be too small), which are in equilibrium, produce the same position in the balance when they are interchanged with each other. On the accurate determination of the equality of the arms and of the sensibility, see (8) and (9).

The following minor points are to be considered in procuring a balance:—The apparatus for moving the rider should be provided with stops to prevent it striking the beam. It should also, as well as the apparatus for stopping the beam, and the doors of the case, have an easy, quiet motion. To avoid parallax in reading, the extremity of the pointer should move very near in front of the divisions, or, still better, above them. As to the size of the divisions, about a millimetre is to be recommended. It is less important that the two scales should be of the same weight, than that the shorter pan provided for specific gravities should be accurately equal in weight to one of the longer ones.

Use of the Balance.—It should stand on a table protected from the tremors of the floor. If it is impossible to help weighing in a heated room, or one into which the sun shines, we must at least protect the balance from unequal heating. To preserve from rust, and to exclude as much as possible the influence of hygroscopic moisture during weighing, a vessel filled with caustic potash or calcium chloride is placed in the balance-case.

Weights must only be put on when the balance is stopped. When the larger weights are to be put on, or when the load is to be taken off the balance, the pans also should be stopped if any apparatus is provided for the purpose. Swinging of

the scales during weighing may give rise to errors. After every weighing with large weights we must make sure that the zero point is unaltered, or make a new determination of it. Any small corrections which may be necessary are made with the foot-screws. It is self-evident that the final weighings are performed with the case shut.

7.—Weighing by observing the Swinging of a Balance.

The ordinary operation of weighing in which weights are added, and at last the rider is moved until the pointer of the balance swings equally on both sides of the middle division, has several defects. First, it requires that the pointer shall point accurately to the middle division when the balance is loaded. It demands, therefore, on account of the unavoidable alteration of the zero point, frequent readjustment of the balance, which takes up much time. In the next place, it is only applicable to balances provided with riders. Thirdly, the process takes a long time, and requires several careful observations, which nevertheless are of no use in determining the result. Finally, it is as a rule better to make a measure depend, not on a trial whether two quantities are equal, for equality is only approximately attainable, but on a trial as to how much they differ.

The following method of weighing, by observing the oscillation of the pointer and by interpolation, escapes these objections. A similar method may be used in many physical measurements, with the same advantage as in the case of the balance—namely, simplification of the means required, greater sensitiveness, and frequently saving of time.

The first thing to be done is the determination of the zero point, by which we mean the point of the scale at which the index comes to rest when the balance is unloaded. Since we cannot and should not wait until the motion ceases to determine this point directly, we must deduce it from observing the division to which the index attains when swinging.

Where we require only moderate accuracy, it is enough to observe two successive points at which the index turns, and to take their arithmetical mean. If we desire greater exactness, and wish to take into account the fact that the amount of oscillation gradually becomes less, we observe several points of turning on both sides, taking care, for the sake of simplifying the reductions, that the first and last shall be on the same side, *i.e.* we make an uneven number of observations. Five or seven are always enough. We then take the arithmetical mean of the observations on the one side, that is, the 1st, 3d, 5th; and of those on the other, viz. the 2d and 4th; and again take the mean of these two numbers. This is the required zero point. In order not to have to distinguish the deviations to the right and left by signs, we call the middle point of the scale not 0 but 10.

Example—

Turning points.					Means.	Zero.
No. 1.	2.	3.	4.	5.		
10·4		10·3		10·3	10·33	9·74
	9·1		9·2		9·15	

Now, place the body on the one scale, and bring the balance nearly to the zero point (within one or two scale divisions) by weights in the other, and at last by moving the rider from one division of the beam to another. Make another set of observations of the swinging as above, then take off or add one or more milligrammes, according as the weights were too heavy or too light, until the position of equilibrium falls on the other side of the zero point, and determine it by again observing the excursions of the index.

The required weight p_0 of the body—*i.e.* the number of weights that must be put on that the balance when loaded may settle to the zero point—is given, by a simple interpolation, from these observations.

Let there have been found

the zero e_0
with the weight P the position E
with the weight p the position e,

we have, since for small deflections the difference of the positions of equilibrium is proportional to the difference of the weights—

$$\frac{e_0 - e}{E - e} = \frac{p_0 - p}{P - p}$$

$$\text{therefore } p_0 = p + (P - p)\,\frac{e_0 - e}{E - e}.$$

The above differences must all be taken with the proper sign, on which account it is simpler to have the scale divisions numbered, so that increased reading corresponds to increased weight.

The operation may also be expressed somewhat more simply, thus:—The two observations with different weights give the difference a of position (the deviation), which corresponds to 1 mgr. increase of the weight. If, further, we determine by subtraction the number of divisions A at which the point of equilibrium is from the zero point with one of weights (it is immaterial which, but to simplify the calculation the nearest to the zero is usually chosen), the number of milligrammes which must be added (or subtracted), so that the balance may settle to the zero, is given by division $= \frac{A}{a}$. Compare also the beginning of the next article.

Example.—The value for the zero has been determined above to be 9·74.

Weight. mgr.	Turning point.					Mean.	Point of rest.
3036	7·8		7·8		7·9	7·83	9·04
		10·3		10·2		10·25	
3037	9·5		9·4		9·3	9·40	9·95
		10·5		10·5		10·50	

Deviation for 1 mgr. = 0·91 scale division.

3037 mgr. were accordingly too heavy by

$$\frac{9{\cdot}95 - 9{\cdot}74}{0{\cdot}91} = \frac{0{\cdot}21}{0{\cdot}91} = 0{\cdot}23 \text{ mgr.}$$

$$\text{weight } p_0 = 3036{\cdot}77 \text{ mgr.}$$

Or by the previous formula,

$$p_0 = 3036 + \frac{1 \times 0{\cdot}7}{0{\cdot}91} = 3036{\cdot}77.$$

With a little practice time is saved by this method of observation, since the performance of the reductions soon becomes quite mechanical, whilst the accuracy is greater than in the ordinary method.

The oscillation should amount to between 1 and 4 scale divisions.

It is immaterial whether the weights are reckoned in grammes or milligrammes, but one settled method should be kept to. The recording of the observations should also be kept in a regular form as above.

8.—Determination of the Sensitiveness of a Balance.

By the sensitiveness of a balance we mean the difference of indication for 1 mgr. difference in the weight. The determination of this quantity is important as a criterion of the excellence of the balance, and farther, as a means of simplifying the process of weighing; for if we possess a table in which the change of position for 1 mgr., when different weights are on the balance, is given, it is enough, in addition to determining the zero point, to make one single observation of the position with nearly the right weight.

The method of proceeding is self-evident. The load for which the sensitiveness is to be determined is put into each pan, and into one a small excess, so that the position of equilibrium is from 2 to 4 divisions from the centre. This position is accurately determined according to the method of the last article; we call it e.

Now, by adding a weight of a milligrammes to the other pan, the position of equilibrium is to be brought to about as far on the other side of the centre, and observed as before. If this position be called e', the required sensitiveness is

$$\frac{e - e'}{a}$$

When this quantity has been determined for different loads (with the ordinary balances used in analysis, at intervals of 10 grms.), the results are entered graphically on paper ruled in squares, the load as abscissa, the sensitiveness as ordinate. Through the resulting points draw a curve, which can either be used directly or for the construction of a table for suitable intervals of load.

On the regulation of the sensitiveness see (6).

The dependence of the sensitiveness on the load arises from the relative positions of the middle and end knife-edges. On the grounds of convenience a sensitiveness independent of the load is to be desired in fine balances, which requires the three edges to be in the same plane. But as this condition can, strictly speaking, be fulfilled for only one definite weight, on account of the bending of the beam, the best makers are accustomed to produce it for a mean load. Hence there is at first a slight increase of sensitiveness with increased load, and then for still greater weights a decrease. By "load" is understood that in *one* of the pans.

9.—Determination of the Ratio of the Arms of the Balance.

The two arms of the balance are inversely as the weights, which, when placed in the corresponding pans, bring the balance to the zero point (7). Since, usually, the absolute accuracy of the set of weights cannot be assumed, the following method is used:—

The zero is observed; then in each pan weights are placed of the same nominal value, about equal to half the maximum which the balance will carry, and made equal by adding milligramme weights, or moving the rider until the balance is in equilibrium, in which proceeding we should, for the sake of accuracy, use the method of interpolation (7). Then the weights are interchanged, and again made equal. If we call the two weights p and P, and have found that the balance is in equilibrium when

	Left.	Right.
in the first weighing	$p + l =$	P
in the second weighing	$P =$	$p + r,$

we have, if L and R denote the lengths of the arms of the balance left and right—

$$\frac{R}{L} = 1 + \frac{l - r}{2p}.$$

A small excess of weight on one pan may be considered as a negative weight on the other (see Example 1).

Proof.—According to the law of the lever—

$$L\,(p + l) = R\,.\,P\,.$$
$$L\,.\,P = R\,(p + r),$$

from which, by formulæ 8 and 3, p. 10, we have

$$\frac{R}{L} = \sqrt{\frac{p+l}{p+r}} = \sqrt{\frac{1 + \frac{l}{p}}{1 + \frac{r}{p}}} = 1 + \frac{l - r}{2p}.$$

Example 1.—Balance carries 100 grms. in each pan.

Left.	Right.
(50 grms.)	(20 + 10 + . . .) + 0·83 mgr.
(20 + 10 + . . .)	(50) + 2·56 mgr.

$$l = -\,0{\cdot}83 \qquad r = 2{\cdot}56$$

$$\frac{R}{L} = 1 + \frac{-\,0{\cdot}83 - 2{\cdot}56}{100000} = 1 - 0{\cdot}0000339$$

$$\text{or } \frac{L}{R} = 1{\cdot}0000339.$$

Example 2.—Balance carries 500 grms.

Left.	Right.
(100 + 100 grms.) + 3·3 mgr.	(200)
(200)	(100 + 100) + 0·7 mgr.

$$l = 3{\cdot}3 \qquad r = 0{\cdot}7$$

$$\frac{R}{L} = 1 + \frac{3{\cdot}3 - 0{\cdot}7}{400000} = 1{\cdot}0000065.$$

In the above examples the figures in brackets signify the weights marked with those figures. The zero point is to be determined before and after each weighing on account of the great loads. If any considerable alteration be found, the weighings which are affected are repeated; otherwise the mean of the readings before and after the weighing is taken as the zero. Compare also the remarks on the next section.

From the first determination follows immediately (see 12)—

$$(50) = (20 + 10 + ..) - 0{\cdot}86 \text{ mgr.}$$

From the second—

$$(200) = (100 + 100) + 2{\cdot}0 \text{ mgr.}$$

10.—Absolute Weighing of a Body.

The influence of the inequality of the arms of the balance is eliminated if the apparent weight, as found by weighing, be multiplied by the ratio of the lengths of the arms, using as numerator the length of the arm with which the weights are connected.

Should this ratio be unknown, there are two ways of proceeding:—

1. A double weighing is performed by placing the body to be weighed first on the right-hand pan and then on the left-hand one. If we again call R and L the lengths of the right and left arms of the balance, and p_1 and p_2 the weights which must be placed upon the right and left pans respectively to balance the body; and if we call the required weight P, then

$$PL = Rp_1$$
$$PR = Lp_2$$

$$\text{whence } P = \sqrt{p_1 \, p_2}$$

Instead of the geometrical mean we may use the arithmetical, since p_1 and p_2 only differ very little from each other. (For the proof, see p. 19, No. 3.)

$$P = \frac{p_1 + p_2}{2}.$$

From this we can also immediately find the ratio of the arms

$$\frac{R}{L}=\sqrt{\frac{p_2}{p_1}}=\sqrt{1+\frac{p_2-p_1}{p_1}}=1+\frac{p_2-p_1}{2p_1}.$$

2. *By Taring.*—The body being upon one pan is balanced by loading the other pan in any convenient way; it is then taken away, and weights are put in its place until the former reading of the balance is obtained. The weights put on give the weight of the body.

Taring is simpler, since the zero point is immaterial. In double weighing the influence of errors is lessened by the double observation.

11.—Reduction to the Weight in vacuo.

The object of weighing a body is to determine its mass, *i.e.* its equality with the mass of the weights. The equality of the masses of two bodies of different densities is not given by the equality of their weights unless the weighing is performed *in vacuo.* In the air both the body and the weights lose weight equal to the weight of the air which they respectively displace.

If we call

The apparent weight of the body in air, *i.e.* the weights which balance it in the air $= m$
the density of the air $= \lambda$
($\lambda = 0{\cdot}0012$ as a mean value. See also (18) and Table 6)
the density of the body $= \Delta$
the density of the weights $= \delta$,

the weight *in vacuo* will be

$$M = m\left(1+\frac{\lambda}{\Delta}-\frac{\lambda}{\delta}\right).$$

There is therefore to be added to the apparent weight m a correction $m\left(\frac{\lambda}{\Delta}-\frac{\lambda}{\delta}\right)$, which is so much the greater as the difference

between Δ and δ is greater. It is almost always sufficient to use the mean value 0·0012 for λ. In this case the correction for brass weights may be taken from Table 8.

Proof.—The volume of the body is $V=\frac{M}{\Delta}$, that of the weights $v=\frac{m}{\delta}$. Every body loses in the air the weight of the air which it displaces; the body therefore which we have weighed loses λV, the weights λv. Since the weights, after subtracting these losses, are equal, we have

$$M-\lambda V = m-\lambda v, \text{ or } M\left(1-\frac{\lambda}{\Delta}\right)=m\left(1-\frac{\lambda}{\delta}\right),$$

therefore, on account of the smallness of λ in comparison with Δ or δ, we have, by formula 8 (p. 11)

$$M=m\frac{1-\frac{\lambda}{\delta}}{1-\frac{\lambda}{\Delta}}=m\left(1+\frac{\lambda}{\Delta}-\frac{\lambda}{\delta}\right).$$

Example.—The correction of the apparent weight m of a quantity of water when weighed with brass weights ($\delta=8·4$), amounts to

$m\,.\,0·0012\left(\frac{1}{1}-\frac{1}{8·4}\right)=m\,.\,0·00106$, *i.e.* 1·06 mgr. in every gramme.

Where the question is not of the absolute weights, but only of ratios of weights, as in chemical analysis, the loss of weight of the substance in the air must still be taken account of, though that of the weights may be neglected. (The alteration of the density of the air by alteration of pressure and temperature causes an error, which, when brass weights are used, amounts only in extreme cases to $\frac{1}{100000}$ of the total weight.)

If, for example, a dilute solution of silver be analysed by weighing a quantity of the solution and the silver chloride (density = 5·5) obtained from it; and if P and p be the observed weights, these are reduced to vacuo $P\,(1+0·0012)$ and $p\left(1+\frac{0·0012}{5·5}\right)$. The proportion of silver chloride amounts therefore to

$$\frac{p\,.\left(1+\frac{0·0012}{5·5}\right)}{P\,.\,(1+0·0012)}=\frac{p}{P}\left(1-0·0012\left(1-\frac{1}{5·5}\right)\right)=\frac{p}{P}\times .\,0·999\,.$$

The uncorrected value $\frac{p}{P}$ would therefore be about 0·1 °/₀ too great. The customary neglect of such a simple correction must, in view of the costliness of the balance, the care spent on the weighings, and the very great pretension to accuracy implied in the large number of decimals used, be considered inadmissible.

12.—Table of Corrections for a Set of Weights.

The operation of determining the errors of a set of weights usually depends on the performance of as many weighings as there are weights to be corrected, and on the formation of the same number of equations from them, from which the ratio of the arms of the balance, and that of the weights to each other or to a convenient unit, may be deduced.

In the sets of weights commonly used in analysis, the manner of proceeding is as follows:—

The larger weights are distinguished as

$$50' \quad 20' \quad 10' \quad 10'' \quad 5' \quad 2' \quad 1' \quad 1'' \quad 1'''$$

A double weighing is performed with $50'$ on one side, and the rest of the weights on the other. Suppose it has been found that the balance is in equilibrium, *i.e.* the pointer is in the same position as when the balance is unloaded, when

Left.	Right.
$50'$	$20' + 10' + \ldots + r$ mgr.
$20' + 10' + 10' + \ldots + l$ mgr.	$50'$,

then the ratio of the arms of the balance is (9)

$$\frac{R}{L} = 1 + \frac{l - r}{100{,}000}$$

$$\text{and } 50' = 20' + 10' + \ldots + \frac{r + l}{2}.$$

When $\frac{R}{L}$ has been determined, a single weighing is sufficient for the other weights; for a weight p, on the right pan, is, on account of the length of the arms, reduced to $p\,\frac{R}{L}$ when weighed on the left hand.

Example.—Let $r = -0{\cdot}83 \qquad l = 2{\cdot}53$

$$50' = 20' + 10' + 10'' + 5' + 1' + 1'' + 1''' + 0{\cdot}85 \text{ mgr.}$$

$$\text{and } \frac{R}{L} = 1{\cdot}0000336.$$

Further, if it be found, when comparing $20'$ with $10' + 10''$, that

Left.	Right.
$20' + 0{\cdot}91$ mgr.	$10' + 10''$

keeps the balance in equilibrium, in a balance with equal arms the equal weights would be

$$20' + 0{\cdot}91 \text{ and } (10' + 10'')\,1{\cdot}0000336,$$

$$\text{or } 10' + 10'' + 0{\cdot}67 \text{ mgr.}$$

It follows then that

$$20' = 10' + 10'' - 0{\cdot}24 \text{ mgr.}$$

Suppose that from 5 weighings we have found

$$\begin{aligned} 50' &= 20' + 10' + \ldots + A \\ 20' &= 10' + 10'' \qquad + B \\ 10'' &= 10' \qquad\qquad + C \\ 5' + 2' + 1' + 1'' + 1''' &= 10' + D \end{aligned}$$

where of course A, B, C, D may be either positive or negative.

From these equations the values of the 5 weights must be expressed in terms of some unit—the sum of the single grammes being provisionally considered as one weight. If a comparison with a normal weight be not made at the same time, this unit is so chosen that the correction of the separate weights shall be as small as possible, which is the case when the whole sum is assumed to be correct—*i.e.* when we consider

$$50' + 20' + 10' + \ldots = 100000 \text{ mgr.}$$

Now it is easily found, by first of all expressing all the weights in terms of $10'$, that

$$50' + 20' + 10' + \ldots = 10{\cdot}10' + A + 2B + 4C + 2D = 100000 \text{ mgr.}$$

Calling therefore

$$\frac{A + 2B + 4C + 2D}{10} = S$$

we have

$$\begin{aligned}
10' &= 10000 \text{ mgr.} - S \\
10'' &= 10000 \;\;,, \;\; - S + C \\
5' + 2' + \ldots &= 10000 \;\;,, \;\; - S + D \\
20' &= 20000 \;\;,, \;\; - 2S + B + C \\
50' &= 50000 \;\;,, \;\; - 5S + A + B + 2C + D \\
&= 50000 \;\;,, \;\; - \tfrac{1}{2}A.
\end{aligned}$$

The proof of the correctness of the numerical work is easily found from the above to be that the sum of the corrections, when expressed as numbers, must equal 0, and the 4 equations given above must be fulfilled.

Again, the following equations having been obtained by comparing the weights $5'$, $2'$, $1'$, $1''$, $1'''$ with each other,

$$\begin{aligned}
5' &= 2' + 1' + 1'' + 1''' + a \\
2' &= 1' + 1'' \qquad\qquad + b \\
1'' &= 1' \qquad\qquad\qquad + c \\
1''' &= 1' \qquad\qquad\qquad + d
\end{aligned}$$

As in the previous case calling

$$\frac{a + 2b + 4c + 2d + S - D}{10} = s$$

we have

$$\begin{aligned}
1' &= 1000 \text{ mgr.} - s \\
1'' &= 1000 \;\;,, \;\; - s + c \\
1''' &= 1000 \;\;,, \;\; - s + d \\
2' &= 2000 \;\;,, \;\; - 2s + b + c \\
5' &= 5000 \;\;,, \;\; - 5s + a + b + 2c + d.
\end{aligned}$$

In the same manner we proceed with the smaller weights, only remarking that usually the inequality of the arms of the balance no longer needs consideration.

We have hitherto assumed the sum of the larger weights to be correct, in order to have corrections as small as possible. For most purposes (chemical analysis, specific gravity) which only require *relative* weighings, this assumption may be made. In order to refer the table of errors to an accurate gramme-weight, it is necessary to compare the weights, or one of them, with a normal weight (10, 11). The calculation is easily got from the above.

A similar method of testing a series of weights of any other arrangement will be easily found.

To distinguish the weights of the same nominal value,

the figures should be differently engraved, or they should be provided with distinguishing marks, otherwise accidental marks must be looked for.

In the case of weights which consist of pieces of foil, the turning up of different corners may be made use of. No regard need be paid to the loss of weight from weighing in the air, for the larger weights are all of the same material, and with the smaller ones the difference is without noticeable influence. For testing the smaller pieces a lighter balance is, when possible, made use of, *i.e.* one which is more sensitive, with the same time of oscillation. The weighings are made by observations of the swing after (7), and the observation of the zero point should be frequently repeated. It is customary to use the weights in a fixed order, so that each total weight will always be made up of the same individual weights; it is easy therefore to calculate the table of errors for the total weights by taking it for every 10000, 1000, 100, 10, and eventually for every milligramme.

13.—Density or Specific Gravity.

By the density or specific gravity of a solid or liquid (we shall call this Δ; see Tables 1 and 2) is meant the ratio of its mass to that of an equal volume of water at 4°. This latter has therefore the density 1. Instead of the ratio of the masses we may use that of the weights *in vacuo*. If the metre and gramme system of weights and measures be used, we may call the density the ratio of the weight to the volume, or, in the case of homogeneous bodies, the weight of unit-volume. In this case, mgr. and mm., gr. and cm., kgr. and dm., naturally belong to each other.

By the density of a gas, according to this definition, is usually meant that which it has at 0°, and under a pressure of 760 mm. of mercury. But most frequently a gas is compared not with water but with dry atmospheric air of the same temperature and under the same pressure.

The methods of determining densities, considering them at present without corrections (for which see sections 14 and 15), are the following:—

For Liquids.

1. Weighing a volume measured in a graduated vessel, such as a tube or pipette. On account of capillary attraction the volume in a graduated tube should be measured by an observation of difference; always reading off the position of the horizontal (upper or lower) surface. To avoid parallax, the necessary observations are made with a telescope sliding on a vertical stand; or, more simply, by always taking one and the same distant point as that to which to direct the eye.

2. The quantity m of the fluid is weighed, and also the quantity w of water which is contained in one and the same vessel (specific gravity bottle, pyknometer). Then we have $\Delta = \frac{m}{w}$.

3. A body (*e.g.* a piece of glass) is weighed in the air, in the liquid, and in water. If we find the loss of weight in the liquid to be m, that in water w, we have again $\Delta = \frac{m}{w}$. Mohr's balance is very simple and convenient, using a rider equal in weight to the water which the body displaces.

4. Scale areometers (hydrometers) give by the division to which they sink either the density or the volume, *i.e.* the reciprocal of the density; or, in the older scales, the so-called "degrees of density." For the relation of these scales see Table 3.

5. The heights of columns of different fluids in tubes communicating with each other are, when equilibrium is established, in the inverse ratio of the densities.

For Solids.

1. Weighing and measuring the volume.—The measuring, when the body is of a regular shape, may be done by a scale; when the body is irregular the volume may be measured by observing how much the surface of a quantity of liquid contained in a graduated tube rises when the body is put into it. This method is specially applicable to substances in small pieces. For substances soluble in water, some other fluid, *e.g.* a saturated solution of the substance, may be used.

2. If m be the weight of the body, and it lose, when weighed in water, the weight w, $\Delta = \frac{m}{w}$.

The body is usually hung to one of the pans of the balance by a thread or wire as thin as possible. The weight of the wire is previously determined, and allowed for in the calculation, in a manner easily seen. The loss of weight of the wire is to be subtracted from w. This can easily be obtained by calculating the weight of the immersed part of the wire from the ratio of the immersed part to the whole length, and dividing by the density of the wire (Table 1).

When weighing in water the oscillations quickly decrease; the position is observed when the balance has come to rest. The thread should only cut the surface of the water in one place, in order not to increase the capillary attraction, which otherwise would impair the accuracy of the weighing.

Instead of hanging the body to the pan of the balance, a vessel of water may be placed upon it, and the increase of weight determined when the body is suspended in it by a thread from a fixed stand. This increase is equal to the apparent loss of weight of the body in water.

With Nicholson's hydrometer the weight in air and in water is determined by the difference in the weights which must be put on in order to sink the instrument to a mark on the stem. Changes of temperature impair the accuracy more when the body is small compared to the hydrometer. Rubbing the stem with spirits of wine makes the certainty of the adjustment greater.

A spiral wire (piano wire) with two pans hung one above the other, one always immersed in a vessel of water, is very convenient for determining densities, especially of small bodies—(*Jolly*). We observe, exactly as with the hydrometer, the weights which must be put upon the upper pan to bring a mark upon the lower end of the wire to the same position when (1) the pans are empty, (2) the body is upon the upper one, (3) it is upon the lower. As fixed index, a mark upon a piece of looking-glass may be used to avoid parallax

If the body must not be put into water it is weighed in some other fluid of known density. The result, calculated as above, must then be multiplied by this density.

When the body is specifically lighter than water it must be made to sink by fixing to it another sufficient weight, *e.g.* a metal clamp or a net of wire gauze, under which the body is allowed to ascend.

3. The weight of the volume of water equal to that of the body can be determined by the specific gravity bottle. If it weigh P when quite full of water, P' when the body has been put in and the displaced water removed, and if m be the weight of the body, $w = P + m - P'$. This method is specially applicable in the case of small bodies, but then flasks as small as possible should be used. As to the corrections see next article.

In every case the air-bubbles, which easily adhere to the bodies, must be removed either by repeatedly dipping in and taking them out again, or by the application of a brush.

14.—Determination of Density by the Specific Gravity Bottle.

One of the most elegant means of determining densities is the specific gravity bottle (pyknometer). Whenever only a small quantity of the substance can be obtained it is the only method available, but it then requires great care on account of the expansion of water with the temperature. The weight of water which the flask would contain at any given temperature can be calculated in the following manner from the weight, taken once for all, of the bottle when full of water at any one temperature.

Let us call the temperature and density of the water at the time the weighing was performed t and Q (Table 4), the weight of water p, and the corresponding quantities for another temperature t', Q', p'. These last quantities are to be calculated.

1. If only the most considerable correction—viz. that

depending upon the expansion of water—is to be considered, we have

$$p' = p\,\frac{Q'}{Q}.$$

2. If we have regard to the expansion of the flask, we know that the volume is greater in the proportion $1 + 3\beta\,(t' - t)$, where 3β denotes the coefficient of cubical expansion of the glass. This may usually be taken as $\frac{1}{40000}$. We have therefore

$$p' = p\,\{1 + 3\beta\,(t' - t)\}\,\frac{Q'}{Q}.$$

3. These directions have special importance in determinations of the density of small solids, since by not applying the corrections we should be led to altogether erroneous results. The apparent weight w of a volume of water equal to that of the body, can in most cases be deduced with sufficient accuracy by the following formula:—

$$w = m + P - P' + (P - \pi)\,[Q' - Q + 3\beta\,(t' - t)]$$

In this formula

m = the weight of the body in air;
P = the weight of the bottle when full of water;
P' = the weight of the bottle with the substance, and filled up with water;
π = the weight of the empty flask (need only be approximate).

Further, the temperature and density of the water are—

t, Q at the weighing with water alone;
t', Q', „ with water and the substance;
3β = the coefficient of cubical expansion of the glass.

Proof.—It was shown above that, if p and p' be the weights of the water at t and t', $p' = p\,\{1 + 3\beta\,(t' - t')\}\,\frac{Q'}{Q}$. This expression can be simplified by bearing in mind that 3β, the coefficient of cubical expansion of the glass, is always a very small number; and further, that Q' and Q only differ very little from 1. For by writing $1 + (Q' - 1)$ for Q', and $1 + (Q - 1)$ for Q, we obtain by formula 8 (p. 11)—

$$\frac{Q'}{Q}=\frac{1+(Q'-1)}{1+(Q-1)}=1+(Q'-Q).$$

By formula 7, therefore, the above expression becomes

$$p'=p\,[1+3\beta\,(t'-t)+Q'-Q]=p+p\,[3\beta\,(t'-t)+Q'-Q].$$

In order therefore to calculate from the weight P of the bottle filled with water at the temperature t, that at t', we must add to P the weight of the water $P-\pi$, multiplied by $3\beta\,(t'-t)+Q'-Q$. The glass and water would therefore, at the temperature t', weigh

$$P+(P-\pi)\,[3\beta\,(t'-t)+Q'-Q].$$

But when the body, of the weight m, has been introduced into the vessel, the weight w of water has overflowed, and the whole now weighs P'. Obviously therefore

$$P'+w=P+(P-\pi)\,[3\beta\,(t'-t)+Q'-Q]+m,$$

from which the expression previously given follows.

It will be seen at once that the weight π of the empty vessel need only be determined approximately, for it only occurs multiplied by a factor of the dimensions of a correction.

15.—Density. Reduction to the Weight in vacuo.

The methods for determining densities given in (13), under 2 and 3, require a correction, which is applied according to the following general rule:—

We must first take account of the fact that the water is usually at some other temperature than $+4^\circ$, and therefore has not the density 1. The true density Q is found from the temperature by the help of Table 4 in the Appendix. In the second place, the weighings are to be reduced to weighings *in vacuo*. Table 6 gives the density λ of dry air for all temperatures and pressures which are likely to occur. For the calculation see (18). It is mostly enough to take the mean value $\lambda=0{\cdot}0012$, for the error thus introduced will rarely influence the third place of decimals to the extent of 1. The neglect of the expansion of the water may affect the result to the extent of $\frac{1}{3}$ per cent, the loss of weight in air about 2 in the second place of decimals. No notice need be taken of the inequality of the arms of the balance, provided

that we always place the body upon the same pan, nor of the air displaced by the weights.

We will call

Q the density of the water employed;
λ the density of the air at the time of weighing (compared with water);
m the apparent, *i.e.* uncorrected, weight of the body in air, or, in the case of a fluid, the apparent loss of weight of a body immersed in it;
w the apparent weight of the volume of water of density Q equal to the volume of the body.

As w we may therefore have—

1. With solids, the apparent loss of weight when the body is weighed in water after the method of Archimedes, with a balance or areometer; or the weight of the water displaced by the body when a specific gravity bottle is used.

2. With fluids, the apparent weight of the water in the specific gravity bottle, or of the water displaced by the piece of glass which is weighed in the fluid and in water.

Then the specific gravity reduced to water at 4°, and freed from the influence of the displaced air, is—

$$\Delta = \frac{m}{w}(Q - \lambda) + \lambda.$$

$\frac{m}{w}$ is the rough uncorrected specific gravity.

It will be seen that the influence of the loss of weight in the air vanishes when the density is 1, and becomes larger with the increase or diminution of density of the body, reaching in the case of platinum $\left(\frac{m}{w} = 21\right)$ the value 0·024. If, besides, the expansion of the water by temperature be neglected, the result may be too great by about 8 in the second decimal place.

Proof.—If the body, solid or fluid, have the weight m in the air, and displace the quantity of air l, it would, *in vacuo*, weigh $m + l$. In considering the determination of w we may distinguish three cases. If we have determined the weight of an equal volume of water by weighing, the weight *in vacuo* is $w + l$. If we have

measured the apparent loss of weight w of a solid by immersion in water, this must be increased by l, since the weight *in vacuo* would be so much greater than in the air. In the same way, thirdly, if we have determined the density of a fluid by finding the apparent loss of weight of the same body when weighed successively in water and the fluid, each of these must be increased by l.

If, however, the water have not the temperature of $+ 4°$, but some other, at which its density is Q (Table 2), the same volume of water at 4° would weigh $\frac{w + l}{Q}$. In all cases, therefore, the true density Δ of the body is obtained by the formula—

$$\Delta = \frac{m + l}{w + l} Q.$$

But since $\frac{w + l}{Q}$ is the volume of the displaced air, calling its density (compared with water) $= \lambda$

$$l = \frac{w + l}{Q} \lambda, \text{ or } l = \frac{w\lambda}{Q - \lambda},$$

and substituting this value in the above expression we obtain

$$\Delta = \frac{m}{w} (Q - \lambda) + \lambda.$$

Example.—Suppose a piece of silver weighs

	mgr.
in air	$m = 24312$
in water at 19°·2	$= 21916$
the apparent loss of weight in water, $w =$	2396

The uncorrected specific gravity will therefore be—

$$\frac{m}{w} = \frac{24312}{2396} = 10{\cdot}147.$$

We obtain the corrected value, taking $Q = 0{\cdot}99843$ for 19°·2, from Table 2—

$$\Delta = 10{\cdot}147\ (0{\cdot}99843 - 0{\cdot}0012) + 0{\cdot}0012 = 10{\cdot}120.$$

It is convenient for the working out of the correction, in case logarithms are not used, to subtract Q from 1, and insert the difference δ, always a very small number, in the formula, writing it—

$$\Delta = \frac{m}{w} - (\delta + \lambda) \frac{m}{w} + \lambda.$$

Thus, in the example given above—

$$\Delta = 10{\cdot}147 - 0{\cdot}00277 \times 10{\cdot}147 + 0{\cdot}0012 = 10{\cdot}120.$$

By this means the calculations may be performed mentally.

16.—Density. Reduction to a Normal Temperature.

Δ is the density of the body at the temperature t, which it had at the time of weighing, compared with water at 4°. For a solid body of which the loss of weight in water has been determined, it is, of course, the temperature of the water employed which must be used. From this the density Δ_0 at any other temperature t_0 is found, with the aid of the coefficient of cubical expansion 3β (Table 9), by multiplying by $1 + 3\beta\ (t - t_0)$. It is usual to give the density for 0°, and therefore—

$$\Delta_0 = \Delta\ (1 + 3\beta t).$$

Most fluids have an irregular expansion, which must be taken from special tables. In these the volumes v_0 and v of the same weight of fluid at t_0 and t are found, and then—

$$\Delta_0 = \Delta\,\frac{v}{v_t}\ \text{(Alcoholometry)}.$$

17.—Density. Determination with the Volumenometer.

The object of this instrument is the measurement of the volume of a body which must not be immersed in water, by means of determining the volume of an enclosed quantity of air according to Mariotte's law.

Let the volume of the quantity of air which is to be determined be V, shut off at the atmospheric pressure of H millimètres of mercury (height of the barometer). If this measured volume V be increased by v cc. without any addition of air, and the diminution of pressure h mm. be observed, we have

$$VH = (V + v)\ (H - h),$$

$$\text{and therefore } V = v\,\frac{H - h}{h}.$$

If, on the other hand, V be diminished, and an increase of pressure h be observed—

$$V = v\frac{H+h}{h}.$$

When the volume of the empty vessel has been found, the body is placed in it and the same process is gone through. The difference of the values found is the volume of the body; the density is therefore the weight (in grammes) divided by this difference.

The smaller v and h are compared with V and H, the greater is the influence of the errors of observation on the result. Any alteration of the temperature of the enclosed air by neighbouring bodies, etc., must be avoided during the experiment.

18.—Calculation of the Density of the Air or of a Gas from its Pressure and Temperature.

Let d_0 be the density (as compared with water) under the pressure of 760 mm. of mercury and at 0° (Table 1); then for the pressure b and temperature t, the density d is, according to Mariotte's and Gay Lussac's law—

$$d = \frac{d_0}{1 + 0{\cdot}003665t} \cdot \frac{b}{760}.$$

For a table of the values of the expressions

$$1 + 0{\cdot}003665t \text{ and } \frac{b}{760} \text{ see Table 7.}$$

The density of dry atmospheric air for 760 mm., and 0° in latitude 45° (20) is

$$\lambda_0 = 0{\cdot}0012928.$$

The temperature t and pressure b correspond therefore to the density

$$\lambda = \frac{0{\cdot}0012928}{1 + 0{\cdot}003665t} \cdot \frac{b}{760}.$$

Table 6 is calculated from this formula for convenience of reduction.

For the accurate determination of λ, a knowledge of the humidity of the air is necessary. The density of water vapour is $\frac{5}{8}$ that of air at the same temperature and pressure. If then the tension e (the pressure) of the water vapour in the air be known (28), $\frac{3e}{8}$ must be subtracted from the observed height of the barometer, and the value so corrected used in Table 6 or in the formula given above.

Failing the knowledge of e, the air may be considered to be on the average half saturated with aqueous vapour. This assumption is very nearly made, at least for ordinary temperatures, by taking for b the undiminished height of the barometer, but as the factor of t, 0·004 instead of 0·003665. The humidity may influence λ to the extent of about 1 per cent. 0·003665 is nearly equal to $\frac{11}{3000}$, or $\frac{1}{273}$.

19.—Determination of the Density of a Vapour or Gas.

The density of a vapour (or gas) is usually compared with that of dry atmospheric air at the same temperature and pressure.

The vapour-density of a chemical compound of known composition is calculated by dividing its atomic weight by 28·88. Thus water ($= H_2O$) has the atomic weight 18; its vapour-density therefore is

$$\frac{18}{28 \cdot 88} = 0 \cdot 623.$$

A. *Determination of Vapour-density by the method of Dumas.*

A glass balloon of from $\frac{1}{8}$ to $\frac{1}{2}$ litre capacity, with a glass tube melted into it, which has been drawn out into a point of about 1 mm. diameter, is weighed; then a few grammes of the fluid, the vapour-density of which is to be determined, are introduced into the balloon, which then is to be put, with a thermometer close to it, into a bath of some

liquid, so that the point reaches out, and heat applied until the fluid in the balloon boils. When this is turned into vapour, the heat must be raised at least 10° above the boiling point, and then the point of the balloon is closed with the blowpipe flame. The temperature of the bath and the height of the barometer must be read off at this instant. Then the cooled and well-cleansed balloon is again weighed, observing the height of the barometer and the temperature of the air in the balance-case. Lastly, the point of the balloon is held under water, from which the air has been removed by boiling or by the use of the air-pump; a file-mark is made on it, and it is broken off, the water then rises into the balloon. The filled balloon, and the point which has been broken off, are again weighed. (See III.)

We call

(1.) m, the weight of the balloon when full of air;
(2.) m' „ „ „ vapour;
(3.) M „ „ „ water;
(4.) t and b, the temperature of the vapour, and the height of the barometer at the moment of sealing.
(5.) t' and b', the temperature in the balance-case and the height of the barometer at the weighing with the vapour. If the tension e of the aqueous vapour in the balance-room be observed (28), $\frac{3}{8}$ of its value must be subtracted from b' (but not from b).
(6.) λ' the density of the air, which may be determined from t' and b' by the foregoing article, or taken from Table 6.

I. *Approximate Formula.*—The vapour-density is

$$d = \left(\frac{m' - m}{M - m}\frac{1}{\lambda'} + 1\right)\frac{b'}{b}\frac{1 + 0{\cdot}003665{\cdot}t}{1 + 0{\cdot}003665t'}.$$

Proof.—The weight of the water which fills the balloon or its volume is found from (1) and (3). $V = M - m$. The weight of the vapour D is found from (1) and (2), for their difference is the weight of the vapour, minus the weight L of an equal volume of air, $D - L = m' - m$.

Since now, if δ be the density of the vapour, and λ' that of the air (both as compared with water), $D = \delta\,(M - m)$ and $L = \lambda'\,(M - m)$, the previous formula becomes $(\delta - \lambda')\,(M - m) = m' - m$, and therefore

$$\delta = \frac{m' - m}{M - m} + \lambda'.$$

Finally, the vapour-density d is to be compared with that of air at the temperature t and pressure b, which obtained at the time of sealing the balloon. For this purpose the value of δ given above must be divided by the density λ of the air for $t\,b$. We find then

$$d = \left(\frac{m - m}{M - m} + \lambda'\right)\frac{1}{\lambda},$$

from which we get the formula given above, remembering that

$$\frac{\lambda'}{\lambda} = \frac{b'}{b}\,\frac{1 + 0{\cdot}003665\,t}{1 + 0{\cdot}003665\,t}.$$

II. *Accurate Formula.*—Regard is had (1) to the expansion of the glass, (2) to the expansion of the water with the temperature, (3) to the loss of weight of the water when weighed in air. (We neglect (1) the change of the loss of weight of the balloon and weights with the change of temperature and pressure; (2) that the drop of the fluid which remains in the balloon has a density differing from that of water.)

In addition to the notation above (1) to (6)—

(7.) Q = the density of the water used for weighing (Table 4).
(8.) 3β = the coefficient of cubical expansion of the glass; average $3\beta = \frac{1}{40000}$.

We have—

$$d = \left(\frac{m' - m}{M - m}\,\frac{Q - \lambda'}{\lambda'} + 1\right)\{1 - 3\beta\,(t - t')\}\,\frac{b'}{b}\,\frac{1 + 0{\cdot}003665\,t}{1 + 0{\cdot}003665\,t'}.$$

Proof.—From the apparent weight of the water $M - m$ (volume V), the weight reduced to *vacuo* is obtained by adding $V'\lambda'$, the weight of the displaced air. The water has the density Q, therefore the weight of water at 4°, which fills the balloon, that is, the volume of this latter, is $V' = \frac{M - m + V'}{Q}$, from which $V' = \frac{M - m}{Q - \lambda'}$. Therefore, as above, we find the weight of the vapour—

$$D = m' - m + V'\lambda' = m' - m + \frac{M - m}{Q - \lambda'} \cdot \lambda'.$$

This vapour has at the temperature t, at the moment of sealing, the volume V, which the balloon has at that temperature—

$$V = \frac{M - m}{Q - \lambda'} \{1 + 3\beta (t - t')\}.$$

From which we find the density δ of the vapour, compared with water (formula 4, p. 10)—

$$\delta = \frac{D}{V} = \left\{ \frac{m' - m}{M - m} (Q - \lambda) + \lambda' \right\} \{1 - 3\beta (t - t')\}.$$

The vapour-density δ, compared with air of the density λ for b, t, is therefore

$$d = \left\{ \frac{m' - m}{M - m} (Q - \lambda) + \lambda' \right\} \{1 - 3\beta (t - t')\} \frac{1}{\lambda},$$

for which, as before, the formula given at first may be put

III. It frequently happens that the atmospheric air is not completely expelled by the boiling of the substance in the balloon, which is known by the balloon not becoming completely full when the point is broken under water. If we do not intend to take account of this, the globe must be filled up with the wash bottle before the weighing, and the calculation proceeded with after the preceding formulæ. The error will be greater the more the density of the vapour differs from 1. Otherwise the balloon must, after breaking off the point, be immersed so far that the inner and outer surfaces stand at the same height (the bubble was sealed in under the atmospheric pressure), and weighed filled to that extent. Then the rest is filled with water, and the weight M determined. We will put

(9.) The weight of the balloon partially filled with water $= M'$.

Then the vapour-density is—

$$d_{\circ} = \frac{(m' - m) \dfrac{Q}{\lambda'} + M' - m'}{(M - m) \dfrac{b}{b'} \dfrac{1 + 0{\cdot}003665t'}{1 + 0{\cdot}003665t} \{1 + 3\beta (t - t')\} - (M - M')}.$$

Proof.—The volume of the included air-bubble, from the weights M and M', is, at the time of filling, $=\dfrac{M-M'}{Q-\lambda'}$; it was therefore at the time of sealing

$$v=\frac{M-M'}{Q-\lambda}\cdot\frac{b'}{b}\,\frac{1+0{\cdot}003665\,.\,t}{1+0{\cdot}003665\,.\,t'}.$$

The expression for d calculated above is therefore the density of a mixture of the volume v of air, and $V-v$ of the vapour; and if we call the density of the pure vapour d_0,

$$Vd=v+(V-v)\,d_0,$$

from which

$$d_0=\frac{Vd-v}{V-v}=\frac{d-\dfrac{v}{V}}{1-\dfrac{v}{V}}.$$

Here for d we substitute the value found by II., and for $\dfrac{v}{V}$

$$\frac{v}{V}=\frac{M-M'}{M-\;m}\,\frac{1\;+\;0.003665\,.\,t}{1\;-\;0{\cdot}003665\,.\,t'}\,\frac{b'}{b}\,\{1-3\beta\,(t-t')\},$$

in which the two last factors may usually be neglected.

After some transformations, with the aid of the approximation formulæ p, the formula given first easily follows.

Example.—We will calculate an example by the formulæ given above, in order to show the magnitudes of the errors to which they lead, and we will take one in which the errors hold about an average ratio.

Let the data given by observation be (the weights being expressed in grammes)

$m\;=68{\cdot}4522$ (air) $\qquad M=293{\cdot}91$ (full of water);
$m'=68{\cdot}7863$ (vapour) $\qquad M=291{\cdot}73$ (partly full of water).

Let the height of the barometer and temperature be

$b=745{\cdot}6$ mm. $\qquad t=105^\circ{\cdot}5$ (at the time of sealing);
$b'=742{\cdot}2$ mm. $\qquad t'=18^\circ{\cdot}7$ (at the time of weighing of the vapour).

Let the tension of the atmospheric vapour at the latter operation be $e=9{\cdot}4$ mm. (28).

The temperature of the water weighed $=17^\circ{\cdot}4$, to which corresponds (Table 4) $Q=0{\cdot}99877$.

We find (18) $\lambda' = 0{\cdot}0011818$ (neglecting e);
$\lambda' = 0{\cdot}0011762$ (regarding e).

The correct value calculated by III., taking account of e, is 2·918. II. gives 2·894; I. 2·904. Neglecting e we obtain 2·925, 2·901, and 2·911.

From this we see that in our example the third decimal is in error to the extent of + 7 on account of neglect of the humidity; by − 24 on account of the air remaining in the globe (here 2·2 c.c. in a total of 225 c.c.); by + 10 from neglecting the expansion of the water and of the balloon, and the loss of weight of the former when weighed in air.

The expression $1 + 0{\cdot}003665t$, which occurs frequently, is found in Table 7. If this be not used, we may use $\frac{272{\cdot}8 + t'}{272{\cdot}8 + t}$ instead of $\frac{1 + 0{\cdot}003665t'}{1 + 0{\cdot}003665t}$.

B. *Gay Lussac's Method (Hofmann).*—A small quantity of the fluid of which the vapour-density is to be determined, is introduced into a thin small bulb of glass, or a very small flask with a ground stopper, and weighed. The bulb and its contents are then placed in a glass tube full of mercury, and inverted in a vessel of mercury. The tube is to be divided into cubic centimetres from the closed end. If the upper end of the tube be now warmed, the bulb bursts, or the stopper is forced out by the pressure of the vapour of the liquid, which becomes vapour above the mercury. The temperature must in every case be raised a few degrees above that at which the whole of the liquid is evaporated.

If now we call

m, the weight of the evaporated substance (in grms.);
v, the volume of the vapour (in cc.);
t, the temperature of the vapour;
b, the height of the barometer in the room;
h, the height of the mercury over which the vapour is above that in the bath;
e, the tension of the vapour of mercury for the temperature t (Table 15);

the desired vapour-density (see beginning of article) is

$$d = \frac{m}{v} \frac{1 + 0{\cdot}003665 \,.\, t}{0{\cdot}001293} \frac{760}{b - h - e},$$

$$\text{or } d = \frac{m}{v\,\lambda},$$

in which λ can be taken from Table 6 for temperature t, and height of barometer $b - h - e$.

C. *Density of a Gas.*—In order to determine the density of a permanent gas, a glass globe with a tube, best with a stop-cock in it, melted on, is filled with the gas by first filling the globe with mercury, inverting in it a mercury-trough, and displacing the mercury by the ascending gas. The globe is closed and weighed (m'). Then the gas is displaced by a sufficient current of air (air of the balance-room, not dried), and the globe weighed open (m). Lastly, weighing the globe filled with water gives the weight M. As above, let b and t represent the height of the barometer and the temperature at the instant of shutting in the gas, where the height of the remaining column of mercury has already been subtracted. t' and b' are the data for the weighing of the globe full of gas. The density of the gas is then calculated according to formula I. or II., pp. 46, 47.

A small quantity of mercury left in when filling with gas is without influence if it is left unaltered in all the weighings.

20.—Determination of the Atmospheric Pressure (Height of the Barometer).

The readings of the barometer require corrections of some importance. The correction, especially depending on the expansion of mercury with the temperature, usually amounts to several millimetres.

(1.) The height of the barometer is given by the height of a column of mercury at 0°, which is held *in equilibrio* by gravity and the pressure of the air. Mercury expands 0·000181 of its volume for each degree of temperature. Therefore, if l be the height of the barometer as read off at the temperature t, its value b, reduced to 0°, is—

$$b = l - 0{\cdot}000181 \,.\, l \,.\, t.$$

It is frequently sufficient to use for l in the correction member a mean value, and perform the correction by subtracting $0{\cdot}135\,.\,t$ mm.

(2.) On account of the expansion of the scale, the length of this also must, in accurate measurements, be reduced to its normal temperature t_0, by the addition of $\beta\,(t-t_0)\,l$, where β denotes the coefficient of expansion (lineal) of the material of the scale (0·000019 for brass, 0·000008 for glass). If, as is usually the case, the normal temperature be 0°, the height of the barometer completely corrected for temperature becomes—

$$b = l - (0{\cdot}000181 - \beta)\,lt.$$

The correction amounts therefore,

for a brass scale to $-\,0.000162\,.\,l\,.\,t$;
for a glass scale to $-\,0{\cdot}000173\,.\,l\,.\,t$.

(3.) In order to correct a cistern barometer for capillary depression, we must add to the observed height the value δ, taken from Table 16 for the interior radius r of the tube.

The comparison with a normal barometer, of course, requires these corrections in its result.

(4.) At high temperatures the tension of the mercury-vapour occasions a slight depression, which is corrected accurately enough (Table 15) by adding $0{\cdot}002t$ mm. to the observed height.

(5.) By the foregoing corrections the true height of the barometer is obtained. For many purposes, however, the knowledge of the pressure of the air is desired, and in this case it must be remembered that the pressure of the air is proportional to the height of the barometer, only under the condition that the force of gravity remains constant. As the normal force that, g_0, is usually taken, which is found at the level of the sea in the geographical latitude 45°. If we call the force of gravity in latitude φ, and at the height H above the sea level g, we have—

$$\frac{g}{g_0} = 1 - 0{\cdot}0026\,.\,\cos 2\varphi - 0{\cdot}0000002\,.\,H.$$

We must therefore multiply the observed height of the barometer by this expression, of which the last member becomes of any importance only at very considerable heights, in order to obtain that which corresponds to the same elasticity of the air in latitude 45° at the sea level.

21.—Hypsometry—Measurement of Heights by the Barometer.

If the height of the barometer be observed at the same time at two different stations, or if the mean height of the barometer at each be known, the difference in height of the stations may be obtained by the following rules. We denote by

b_0 and b_1 the two barometer readings reduced to 0°, and, if necessary, corrected for capillary depression and the tension of the mercury vapour (previous article);
t_0 and t_1 the temperature of the air at the two stations;
h the required difference of height in metres;

and for convenience calling, further,

t = the mean temperature of the air between the two places,—

therefore $t = \frac{1}{2}(t_0 + t_1)$.

I. It is usually reckoned that

$$h = 18420 \text{ met. } (\log b_0 - \log b_1)(1 + 0{\cdot}0039 \,.\, t),$$

from which, for differences of height not exceeding 1000 metres, we may obtain the convenient approximation—

$$h = 16000 \text{ met. } \frac{b_0 - b_1}{b_0 + b_1}(1 + 0{\cdot}0039 \,.\, t).$$

II. If the variation of gravity on the earth's surface be taken into consideration, we assume further,

φ the latitude,
H the mean height of the two places above the sea-level in metres. For this it is enough to use a rough approximation, within 500 metres.

Then

$$h = 18420 \text{ met. } (\log b_0 - \log b_1)\,(1 + 0{\cdot}0039\, t)\,.\,(1 + 0{\cdot}0026 \cos 2\varphi + 0{\cdot}0000002\, H).$$

III. In the above formula a mean amount of moisture in the air is assumed, but if, at the same time as the barometer is read, an observation be made with the hygrometer or psychrometer (28) at each station, we may take

e_0 and e_1, the tension of aqueous vapour at the two stations;

and for shortness—

$$k = \tfrac{1}{2}\left(\frac{e_0}{b_0} + \frac{e_1}{b_1}\right),$$

and calculate the difference of height from the formula—

$$h = 18405 \text{ met. } (\log b_0 - \log b_1)\,(1 + 0{\cdot}0039\, t)\,.\,(1 + 0{\cdot}0026\,.\cos 2\varphi + 0{\cdot}0000002\, H + \tfrac{3}{8}\, k).$$

The logarithms in this formula are the common Briggs's logarithms.

For convenience of carrying, the height of the barometer in measurement of heights is frequently deduced from the boiling-point of water. Tables 10 and 11 give the corresponding boiling-points and pressures. Since 1 mm. of pressure corresponds to $\frac{1}{25}$ of a degree, it follows that very sensitive, accurately-verified thermometers, as well as the greatest care, must be employed in the temperature determination (22) if we wish to arrive at a tolerably accurate result.

Proof of the hypsometric formula.—The density of atmospheric air (18 and 20) in latitude φ, the height H, with the height of barometer b, the temperature t, and the tension e of the aqueous vapour; calling, for shortness, $0{\cdot}0026\,.\cos 2\varphi = \delta$, $0{\cdot}0000002 = \epsilon$, and $0{\cdot}003665 = \alpha$, is

$$\frac{0{\cdot}0012928}{1 + \alpha t}\,.\,\frac{b - \frac{3}{8}e}{760}\,(1 - \delta - \epsilon H).$$

Now the density of mercury at 0° is 13·596; it follows, if the increase of height dH diminish the height of the barometer b by db

(*i.e.* dH and db are the heights of the columns of air and mercury respectively which are *in equilibrio*)—

$$-db = \frac{0{\cdot}0012928}{13{\cdot}596 . 760}\left(b - \tfrac{3}{8}e\right)\frac{1 - \delta - \epsilon H}{1 + \alpha t}\, dH.$$

Here, besides b, we have e and t varying with H, but according to an unknown law. Hence we take for t the constant mean value, and put e in a constant ratio to the height of the barometer, $e = kb$. If, then, we calculate out the numerical factor, and consider the small quantities $\frac{3}{8}k$, δ, and ϵH, according to p. 10, as corrections, we may write—

$$-7993000\,(1 + \alpha t)\,(1 + \delta + \tfrac{3}{8}k)\,\frac{db}{b} = (1 - \epsilon H)\, dH.$$

Integrating between the limits b_0 and b_1 on the left-hand side, and H_0 and H_1 on the right, we have—

$$7993000\,(1 + \alpha t)\,(1 + \delta + \tfrac{3}{8}k)\,(log.\, b_0 - log.\, b_1) = (H_1 - H_0)\left(1 - \epsilon\,\frac{H_1 + H_0}{2}\right),$$

the logarithms being natural logarithms.

Finally, putting natural $log\ b = 2{\cdot}3026\ log_{10}\ b$, and considering $\epsilon\,\frac{H_1 + H_0}{2} = \epsilon H$ as a correction, we obtain

$$H_1 - H_0 = h = 18405000 \text{ mm. } (log\ b_0 - log\ b_1)\,(1 + \alpha t)\,(1 + \delta + \epsilon H + \tfrac{3}{8}k).$$

The approximation formulæ for unknown humidity are got by assuming the air half saturated, and neglecting the influence of the aqueous-vapour on the density and the coefficient of expansion.

22.—Freezing and Boiling Points of a Thermometer.

In thermometers, as commonly bought, the two fixed points are often very erroneously marked. To determine the freezing-point, the thermometer is plunged into melting snow or clean broken ice. The point to which the column of mercury reaches, corresponds to the temperature 0°.

The column of mercury should be almost entirely covered by the ice. A way must be provided for the water, produced by the melting ice, to escape into a vessel placed below. Vessels with

walls of some bad conducting substance, *e.g.* wood, are to be preferred. The warmer the surrounding air, the more carefully must these precautions be observed.

For the determination of the boiling-point—*i.e.* the division which corresponds to 100°—the thermometer is placed in the steam of water which is boiling vigorously in a vessel either of metal or of glass, with some pieces of metal in it. The temperature of the steam is found from the pressure under which the water boils—*i.e.* from the height of the barometer reduced to 0° (20)—by the aid of Table 10. Without tables, the boiling-point may be determined to within $\frac{1}{100}$ of a degree for any pressure between 715 and 770 mm. by the formula—

$$t = 100° + 0°{\cdot}0375\ (b - 760).$$

The bulb of the thermometer must not dip into the boiling water, but must be about 1 cm. above the surface. Here also the whole column of mercury should be exposed to the steam. The opening for the escape of the steam must be so wide that there is no additional pressure in the vessel. The flame should be kept at some distance from the parts of the vessel which are not in contact with water. A vessel of double form is convenient, in which the steam, after surrounding the thermometer, enters at the top another chamber, and from this escapes at the bottom into the air. In such a vessel the bulb may be farther from the surface of the water than the distance given above.

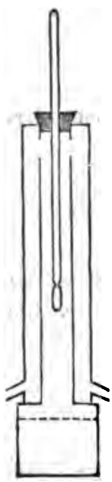


Fig. 1.

In determining both the freezing and boiling points, the reading must not be taken till the observer is convinced that the column of mercury has taken up a permanent position. In exact determinations the reading off is performed with a telescope. The thermometer is adjusted in a vertical position by means of a window-frame or something similar, and the telescope is placed at the height of the division to be read.

Example.—The reduced height of the barometer was 742 mm.

The mercury in the thermometer stood in the steam at 98°. The boiling-point is found from Table 10 to be $99^\circ{\cdot}33$ (from the formula given above $100 - 0{\cdot}0375 . 18 = 99{\cdot}325$). It follows that 100° is denoted by the division $98{\cdot}8 + 0{\cdot}67 = 99{\cdot}47$.

23.—Calibration of a Thermometer.

From the irregular section of the tube there arise in ordinary thermometers errors which, at high temperatures, sometimes amount to more than 10 degrees. We are to prepare, for a thermometer in which only a correct linear division and a scale nearly corresponding to the true temperatures are assumed, a table of corrections, by which the readings can be reduced to those of a normal thermometer—*i.e.* of one of which the 0 and 100 correspond to the freezing and boiling points (see previous article), and of which all the scale-divisions have equal volumes.

In addition to the determinations of the foregoing article, we must therefore undertake the calibration of the thermometer tube—that is to say, compare the volumes which correspond to the divisions of the scale at different places. For this purpose a thread of mercury, separated from the rest, is made use of.

Separation of a Thread of Mercury of any desired length.—The thermometer is turned upside down, and a slight tap given against the end. Then either a thread will separate, or the whole of the mercury will flow down, separating from the walls of the bulb at some point. The separation is usually determined by a microscopical air-bubble adhering to the glass, which expands to a larger size. If the mercury separate in the bulb, we try, by suddenly turning the thermometer upright, to make the bubble formed there rise to the opening of the stem; this can always be done, with patience. The mercury, then, divides at the opening of the tube.

Suppose the thread to be too long, say p degrees longer than was desired. The bulb is warmed while the thread is separated; the air is pushed forward by the rising mercury.

Then the thread is made to run back to the rest of the mercury, and the position of its upper end is observed at the instant of meeting. The little bubble of air remains adhering to the glass at the point of the stem where the junction took place. The thermometer is now cooled p degrees, and again reversed and shaken, when a thread of the desired length is separated.

If, on the other hand, the thread be p degrees too short, it is united to the rest, the thermometer warmed p degrees, when the desired length will break off.

Even if this manipulation should not succeed at first, it always will, on repetition, be possible to get a thread accurately of any length to the fraction of a degree. For very short threads, however, the process often fails; so that, in such a case we must make use, as shown below, of combined observations with threads of different lengths.

Placing and Reading the Thread.—By gentle inclining and shaking, one end of the thread can be adjusted to any desired division with great accuracy. In accurate observations, especially with the telescope, it is sufficient to place it nearly on the division, and estimate the tenths of a degree at both ends of the thread.

Since the thread of mercury and the graduation are not in the same plane, we must avoid parallax when reading off. It is simplest to lay the thermometer upon a piece of looking-glass, and place the eye so that its image coincides with the division to be read; or a lens is fixed steadily, and the thermometer is pushed along parallel to itself under it. The greatest accuracy is secured by reading with the telescope.

Observation and Calculation.—The calibration may be executed in many ways. In every case it is advisable to completely arrange beforehand the plan of the reduction, because afterwards one might be led into complicated calculations. The calculation will always be simplified by making the freezing and boiling points the extremities of compared volumes. Observations, according to the following

plan, are mostly sufficient, instead of methods more exact, but tedious, and requiring long calculations (see Bessel, *Pogg. Ann.*, vol. vi. p. 287); and the more so because completely-corrected mercurial thermometers may differ not inconsiderably on account of the sort of glass of which they are made (see 24, end).

Let a be the interval in which we wish to calibrate, and let a divide 100 without remainder, then $a = \frac{100}{n}$ where n is a whole number. We separate a thread of about this length a; this we place successively at the marks of the graduation, from near 0 to a, a to $2a$, and so on. In each position let the thread occupy the following number of divisions:—

$a + \delta_1$ from the mark 0 to a,
$a + \delta_2$ „ „ a to $2a$,
. . . . .
$a + \delta_n$ „ „ $(n-1)\,a$ to 100.

Let it have been further determined (see previous article)

that the temperature 0° corresponds to p_0;
„ „ 100 „ $100 + p_1$.

The quantities $\delta_1\ \delta_2 \ldots$ as well as p_0 and p_1, are therefore small numbers, expressed in scale-divisions and fractions, and may be either positive or negative.

If, then, we use the abbreviation—

$$\alpha = \frac{p_0 - p_1 + \delta_1 + \delta_2 + \ldots + \delta_n}{n}$$

the correction-table of the thermometer is—

Division.	Correction.
0	$-p_0$
a	$\alpha - p_0 - \delta_1$
$2a$	$2\alpha - p_0 - \delta_1 - \delta_2$
.	. . . .
ma	$m\alpha - p_0 - \delta_1 - \delta_2 - \ldots - \delta_m$.
.	. . . . . .

Or again, the correction for ma being Δ_m, if that for $(m-1)\,a$ be Δ_{m-1}

$$\Delta_m = \Delta_{m-1} + \alpha - \delta_m$$

The values under the heading "Correction" are therefore those numbers which must be added to, or, when negative, subtracted from the corresponding reading, in order to obtain the corresponding reading of an accurate mercurial thermometer.

For the intermediate degrees, a table is interpolated in the usual manner.

Proof.—The thread of mercury used for the observations, laid end to end n times, takes up the volume of the tube from division 0 to 100, increased by $\delta_1 + \delta_2 + \ldots + \delta_n$. But since 0° is at division p_0 and 100° at $100 + p_1$, the increase of the volume of mercury from division 0 to division 100 answers to an increase of temperature of $100 + p_0 - p_1$, so that the increase of the volume equal to the length of the thread means an increase of temperature—

$$\frac{100 + p_0 - p_1 + \delta_1 + \delta_2 + \ldots + \delta_n}{n} = a + \alpha \text{ (see above).}$$

Therefore a rise of the mercury

from 0 to a corresponds to an increase of temperature $a + \alpha - \delta_1$;
" a to $2a$ " " " $a + \alpha - \delta_2$;
. . . . . .

and finally,

from division 0	Temperature increase.
to a	$a + \mid \alpha - \delta_1$;
to $2a$	$2a + \mid 2\alpha - \delta_1 - \delta_2$;
. . . .	.
to ma	$ma + \mid m\alpha - \delta_1 - \delta_2 - \ldots \delta_m$.

The expressions to the right of the stroke would be the thermometer corrections, if the division 0 also meant the temperature 0°. Since the temperature $-p_0$ corresponds to this, p_0 must be subtracted from each of them.

Example.—A thermometer graduated to the boiling-point of

mercury is to be calibrated at intervals of 50°, which is enough for ordinary purposes. Here, therefore, $n = \frac{100}{50} = 2$. A thread of about 50° long was separated, and occupied the spaces—

from	0·0 to 50·9	$\delta_1 = + 0·9$
„	50·0 „ 100·4	$\delta_2 = + 0·4$
„	100·1 „ 150·3	$\delta_3 = + 0·2$
„	149·8 „ 199·8	$\delta_4 = \pm 0·0$
„	200·4 „ 250·0	$\delta_5 = - 0·4$, etc.

In addition, the temperature 0° was found to be at the division + 0·6, and 100° at 99·7; therefore

$$p_0 = + 0·6,\ p_1 = - 0·3.$$

Therefore—

$$a = \frac{p_0 - p_1 + \delta_1 + \delta_2}{n} = \frac{+ 0·6 + 0·3 + 0·9 + 0·4}{2} = + 1·1.$$

The table of corrections is therefore—

Division.		Correction.
0	$- 0·6$	$= - 0·6$
50	$1·1 - 0·6 - 0·9$	$= - 0·4$
100	$2·2 - 0·6 - 0·9 - 0·4$	$= + 0·3$
150	$3·3 - 0·6 - 0·9 - 0·4 - 0·2$	$= + 1·2$
200	$+ 1·2 + 1·1 - 0·0$	$= + 2·3$
250	$+ 2·3 + 1·1 + 0·4$	$= + 3·8$, etc.

The correspondence of the calculated correction for 100 with the determination of the boiling-point furnishes a partial proof of the accuracy of the calculation.

From the last column the correction of any intermediate division is interpolated according to the ordinary rules. For example, to the reading 167·3, the temperature $167·3 + 1·6 = 168°·9$ would correspond.

Calibration by several Threads.—We do not always succeed in separating a thread as short as the interval a with which we wish to calibrate. We must then use several threads, the lengths of which are different multiples of a. By one thread about ka in length we can compare the capacity of the tube between the scale-divisions 0 and a with that between ka and $(k + 1)\,a$, and so on, by bringing the thread first be-

tween 0 and ka, and then between a and $(k + 1)\,a$; for the volume which is left empty by moving the thread is equal to that which is freshly occupied at the other end; the space included in both cases being of no influence. For example, a thread of say 40° long can be used to compare 0 to 20 with 40 to 60.

But in order to reduce all parts to a common measurement, it is obvious that observations must be made with several threads. Two threads of the lengths of $2a$ and $3a$ respectively are always sufficient, for with the first we can reduce 0 to a, $2a$ to $3a$, $4a$ to $5a$, etc., to a common measure; and then the parts not yet compared may be reduced to the same measure by the use of the thread $3a$, by, for example, comparing a to $2a$ with $4a$ to $5a$, etc.

It is scarcely possible to give here any general plan of proceeding, only some rules which should be observed for the sake of convenience and accuracy. Superfluous comparisons mostly lead to minute calculations of equations, which often can be fully carried out with the aid of the method of least squares. They should therefore be avoided, and the same scheme, only making as many comparisons as are necessary, should be repeated in several sets of observations. Further, it is not conducive to accuracy and convenience that single comparisons with the same measure should include many intermediate parts. It is better, therefore, to diminish the number of these by another auxiliary thread. The plan of the reduction, therefore, must be accurately determined in each single case before the observations are made.

In order, now, that we may be able to make use of the plan set forth in p. 59 for the calculation of the table of corrections, it is simplest to deduce from the readings the size which a thread of the length a would assume at the different places. This is provided for in the observations by reducing all the volumes to be compared in as short a way as possible to some one interval, *e.g.* the middle one of all. An example will make this sufficiently clear.

Example.—A thermometer is to be calibrated for every 20

degrees from 0 to 100 by means of two threads of 40° and 60° long. We take the middle part, that from 40 to 60, as the unit volume with which we are to compare the other four. The observations, therefore, are reduced to those numbers which a thread of mercury E, which exactly fills the space from division 40 to 60, would have afforded. According to the above given notation, therefore (p. 59),

$$\delta_3 = 0.$$

Now, let the column of about 40° in two positions occupy the spaces from + 0·3 to 40·0 and 20·7 to 60·0. The column F would therefore have extended from + 0·3 to 20·7 : therefore, $\delta_1 = +\ 0{\cdot}4$.

In just the same say we reduce the space from 80 to 100 to F by observations between 40 and 80, and 60 and 100. Suppose it has been found

$$\delta_5 = -\ 0{\cdot}7.$$

Now, we take a column 60° long, place it between 0 and 60 and 20 and 80. By this we get 60 to 80 in terms of 0 to 20, and, since the latter space has already been compared with 40 to 60, to F. Let the included spaces be—

0 to 60·2, and 20 to 79·6 ;
therefore 0 to 20 = 60·2 to 79·6.

But the column F is longer than 0 to 20 by 0·4 ; it would therefore have extended from 60·2 to 80 ;

$$\text{therefore } \delta_4 = -\ 0{\cdot}2.$$

Finally, in the same manner let observations between 20 and 80 and between 40 and 100 have given—

$$\delta_2 = +\ 0{\cdot}3.$$

It has also been determined that 0° is at + 0·1, and 100° at 100·8 ; therefore—

$$p_0 = +\ 0{\cdot}1 \quad p_1 = +\ 0{\cdot}8.$$

The number of spaces compared between 0 and 100 is $n = 5$. From this we calculate (p. 59)—

$$\alpha = \frac{+\ 0{\cdot}1 - 0{\cdot}8 + 0{\cdot}4 + 0{\cdot}3 + 0{\cdot}0 - 0{\cdot}2 - 0{\cdot}7}{5} = -\ 0{\cdot}18.$$

And the table of corrections is obtained by using the formula, $\Delta_m = \Delta_{m-1} + \alpha - \delta_m$.

Division.	Correction.
0	$-0{\cdot}1$
20	$-0{\cdot}10-0{\cdot}18-0{\cdot}4=-0{\cdot}68$
40	$-0{\cdot}68-0{\cdot}18-0{\cdot}3=-1{\cdot}16$
60	$-1{\cdot}16-0{\cdot}18+0{\cdot}0=-1{\cdot}34$
80	$-1{\cdot}34-0{\cdot}18+0{\cdot}2=-1{\cdot}32$
100	$-1{\cdot}32-0{\cdot}18+0{\cdot}7=-0{\cdot}80.$

The last number is a proof of the correctness of the calculation.

Comparison of two Thermometers.—A table of corrections for a thermometer may also be deduced by comparison at different temperatures with a standard thermometer. The two instruments are placed in a vessel, not too small, filled with fluid, and protected as much as possible from conduction and from radiating heat to the thermometers. It is advantageous to surround the vessel with felt. The bulbs of the two thermometers should be close to each other in the liquid, which should be set in motion by stirring before each reading. At high temperatures the comparison may easily be inexact. (See upon this 27, A.)

The thermometers made by Fastré on Regnault's method are calibrated before dividing, so that each part of the scale corresponds to the same volume. Otherwise the graduation is arbitrary. If the temperature 0° be at p_0 and 100° at p_1, the reading p denotes the temperature

$$\frac{100}{p_1-p_0}(p-p_0).$$

24.—Air Thermometer.

The scientific definition of temperature rests upon the assumption that a perfect gas (*e.g.* dry air) expands, at constant pressure, proportionally to the rise of temperature. The expansion amounts for each degree to 0·003665 of the volume at 0°; or what is identically the same, the pressure of a quantity of air kept at a constant volume increases for each degree of rise of temperature 0·003665 of its pressure at 0°.

The simplest air thermometer (the very convenient form given to it by Jolly) depends upon the latter law. A glass globe of about 50 c.c. capacity, filled with *dry* air, is in communication, by means of a capillary tube, with a vertical glass tube I, in which the air is confined by mercury. By the raising or lowering of the surface of the mercury in II, which is joined to I by an india-rubber tube, the surface of the mercury in I can be brought to a mark near the opening of the capillary tube.

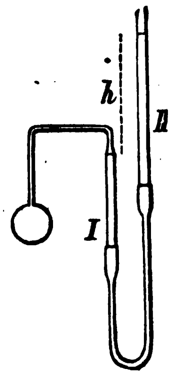


Fig. 2.

To graduate the instrument, the bulb is surrounded with melting ice (see 22), the mercury is adjusted, and the height of the barometer b_0, and the height h_0, of the mercury in II above that in I observed. We will call $b_0 + h_0 = H_0$, where h_0 is negative, when the surface in II is the lower. All the heights b and h must be reduced to 0° by (20).

If, now, any other temperature t which is to be measured be communicated to the air in the bulb, the mercury adjusted to the mark, and the heights b and h be observed, calling $b + h = H$, we have—

$$t = \frac{H - H_0}{0{\cdot}003665\, H_0 - 3\beta H}.$$

3β denotes the coefficient of cubical expansion of the glass. Where this is not known, for the sort of glass used, we may reckon $3\beta = 0{\cdot}000025$. Up to temperatures of about 60° we may calculate it with sufficient accuracy by the more convenient formula—

$$t = 275\,\frac{H - H_0}{H_0}.$$

It is here assumed that the volume of the capillary tube up to the mark to which the mercury is adjusted may be completely neglected in comparison with that of the bulb.

If not, we must add to the value of t, given above, the correction-

$$t \cdot \frac{v'}{v} \cdot \frac{H}{H_0} \cdot \frac{1}{1 + 0{\cdot}003665\, t'},$$

where v = the volume of the bulb, v' that of the connections to the mark, and t' = the temperature of the room.

The ratio $\frac{v'}{v}$ is found by weighing with mercury. If p be the weight of the mercury in the bulb alone, and P the weight when the apparatus is filled up to the mark—

$$\frac{v'}{v} = \frac{P - p}{p}.$$

Proof.—The quantity of air remains constant. If v be the capacity of the bulb at 0°, d_0 the density of the air for 0° and 760 mm., the quantity of air is given at the first observation, if we call $0{\cdot}003665 = \alpha$, by—

$$\frac{d_0 H_0}{760}\left(v + \frac{v'}{1 + \alpha t'}\right),$$

at the second by—

$$\frac{d_0 H}{760}\left[\frac{v\,(1 + 3\beta t)}{1 + \alpha t} + \frac{v'}{1 + \alpha t'}\right].$$

By equating the expressions, dividing by $\frac{d_0}{760}$, and multiplying both sides of the equation by $\frac{1 + \alpha t}{v}$, we get—

$$H_0\,(1 + \alpha t)\left(1 + \frac{v'}{v} \cdot \frac{1}{1 + \alpha t'}\right) = H\left(1 + 3\beta t + \frac{v'}{v} \cdot \frac{1 + \alpha t}{1 + \alpha t'}\right);$$

or separating t—

$$t\left[\alpha H_0 - 3\beta H - \frac{v'}{v}\,\frac{\alpha}{1 + \alpha t'}\,(H - H_0)\right] = (H - H_0)\left(1 + \frac{v'}{v}\,\frac{1}{1 + \alpha t'}\right)$$

From this we get the first of the expressions given above by putting $\frac{v'}{v} = 0$. In order to obtain the correction, we write the left-hand side of the equation—

$$t\,(\alpha H_0 - 3\beta H)\left(1 - \frac{v'}{v} \cdot \frac{\alpha}{1 + \alpha t'} \cdot \frac{H - H_0}{\alpha H_0 - 3\beta H}\right).$$

In the factor of the small magnitude $\frac{v'}{v}$ we may neglect the $3\beta H$, which occurs in the denominator, in comparison with αH_0; and finally we get—

$$t = \frac{H - H_0}{\alpha H_0 - 3\beta H}\left(1 + \frac{v'}{v} \cdot \frac{H}{H_0} \cdot \frac{1}{1 + \alpha t'}\right).$$ (Formula 7, page 11),

as was to be proved.

Comparison of Mercury and Air Thermometers.—Mercury does not expand proportionally to the temperature as measured by an air thermometer. Its volume at the temperature t may be expressed thus—

$$v_t = v_0\,(1 + 0{\cdot}00017905t + 0{\cdot}0000000252t^2),$$

or up to $t = 100$ by—

$$log.\ v_t = log.\ v_0 + 0{\cdot}000078t,$$

an expression which is frequently very convenient. According to this, the readings of the common mercurial thermometer, when they have been corrected after 22 and 23, are between 0° and 100° lower; above 100°, on the contrary, higher than those of an air thermometer, although, on account of the simultaneous expansion of the glass, they are less than would follow from the formula given above. Up to 150° the deviation usually remains smaller than 0°·5; up to 250° it may amount to 4°; up to 350° to 10°. On the average the correction of a mercurial thermometer to an air thermometer may be taken as about—

for reading									
0	20	40	60	80	100	150	200	250	300
correction—									
±0·0	+0·2	+0·3	+0·3	+0·2	±0·0	−0·5	−1·1	−2·4	−3·3

25.—Determination of Temperature with a Thermo-element.

In experiments where the great mass or the circumference of a mercurial thermometer, prevents its use, the electromotive force, developed at the point of contact of two metals (bismuth—antimony; iron—german-silver; platinum—iron) by difference of temperature, may often be made use of. Two wires of equal length (*e.g.* iron and german-silver) are soldered together at one end, and at the other to copper wires. If the former soldering be placed at the point of which the temperature is to be measured, and the other two kept at a known temperature (say by ice at 0°), an electromotive force results. This is measured by connecting the ends of the copper wires with a galvanometer and observing the deflection.

For small differences of temperature (up to about 20°) the current strength may be taken as proportional to the difference of temperature. It is therefore only necessary to measure the current strength for a known difference *once*, in order to deduce the temperature from any observation. A galvanometer, with but few coils of thick wire, reading by a mirror, should be used.

For greater differences, or when the ordinary thermomultiplier is used, in which the current strength cannot be calculated from the deflections, a table is constructed empirically by observing the deflections for certain known temperatures. From this a table for use is interpolated either by calculation or graphically.

Fig. 3.

A convenient form of the thermo-element is the following:—*a* and *b* are the iron and german-silver wires (for use in mercury, iron, and platinum), which are passed through a cork into a small glass tube full of alcohol, within which they are soldered to the copper wires, which are brought through the other cork. In the alcohol a small thermometer can be placed.

26.—Determination of the Coefficient of Expansion by Heat.

By coefficient of expansion is meant the increase of unit length of the body for a rise of temperature of 1°.

I. *By Measuring the Length.*

If the length of a rod $= l$, and if it increase λ in length for a rise of $t°$, the coefficient of expansion $\beta = \frac{\lambda}{2t}$ (see also the example in 3). The small expansions require delicate means of measuring them. If a contact lever be used, and the angle α through which it is turned be measured, $\lambda = r \sin \alpha$, where $r =$ the distance of the point of contact from the axis on which the arm turns, and where also it is assumed that at one of the temperatures the lever arm is perpendicular to the direction of the rod.

The angle is very conveniently measured by observing a scale reflected in a mirror fixed to the lever. We assume that at one of the observations the point at which a perpendicular from the mirror falls upon the scale is seen in the telescope, and that the distance between the scale and the mirror, expressed in scale-divisions as units $= R$. If the motion of the image for the change of temperature amount to n scale-divisions, $\alpha = \frac{1}{2} \tan^{-1} \frac{n}{R}$. Since when α is small we may put $2 \sin \alpha$ for $\tan 2\alpha$, we should have in this case $\lambda = \frac{n}{2} \cdot \frac{r}{R}$. (See also 48.)

II. *By Weighing.*

Very often a need arises of an accurate knowledge of the coefficient of expansion of glass; this can be obtained by a process of weighing. A bulb with a drawn-out point is weighed, filled with mercury at different temperatures. To fill the bulb, it is first warmed and the point plunged into mercury, of which, as the bulb cools, a quantity is drawn up

into it. This is repeated until the bulb is completely full; at last, if necessary, boiling the mercury, plunging the point into warmed mercury, and leaving it there till it has cooled down to the temperature t_0. By weighing, the net weight p_0 of the mercury is obtained. Then it is warmed to the temperature t_1, which causes a certain quantity of mercury to overflow, and the weight p_1 of the remainder is determined. Then the coefficient of expansion (cubical) is calculated thus—

$$3\beta = 0{\cdot}0001815 - \frac{p_0 - p_1}{p_0\,(t_1 - t_0)}.$$

27.—Boiling-Point of a Fluid.

The boiling-point is the temperature of the steam which rises from a fluid boiling under the pressure of 760 mm. of mercury at 0°. The direct readings of the thermometer require two corrections.

(A.) A part of the column of mercury is usually out of the steam. If the length of this be a degrees, t' the mean temperature of the exterior part of the mercury, and t the thermometer reading, we must add to this last—

$$0{\cdot}00016\,.\,a\,(t - t').$$

0·00016 is the difference between the coefficients of cubical expansion of mercury and glass, or the apparent coefficient of expansion of mercury in glass.

As the mean temperature t' of the exterior part of the mercury column is hard to determine, it is taken, in default of anything better, from the reading of a second thermometer, the bulb of which is placed in contact with the stem of the thermometer, used in the determination at about the middle of the exterior part.

(B.) The boiling-point must be reduced to 760 mm. from the actual height of the barometer b at the time of observation. It is indeed only very rarely that the rise of the boiling-point in proportion to the increase of pressure is known, which would be necessary to the accurate correction. But since the boiling-points of most fluids which have been

investigated vary according to nearly the same law in the neighbourhood of 760 mm., on an average—that is, this temperature increases by 0·0375, or $\frac{3}{80}$ of a degree, for 1 mm. increase of pressure—the correction may be applied approximately by adding to the observed temperature

$$0{\cdot}0375 \, . \, (760 - b).$$

The thermometer must not dip into the liquid, but only into the steam. To make the boiling regular, pieces of platinum foil are put into the liquid. (See also 22.)

28.—Determination of the Humidity of the Air (Hygrometry).

The magnitudes to be here determined are—

(1.) The density of the vapour of water in the air—*i.e.* the weight in grammes of the water contained in 1 c.c. of air. Since this number is very small, it is usual to multiply it by 1000000, by which we obtain the weight of the water in 1 cubic metre of air, expressed in grammes. This is called in meteorology the *absolute humidity* of the air. We shall in the rest of this article call it f.

(2.) The *relative humidity*, or the ratio of the amount of water actually existing in the air, to the amount which would saturate it. This quantity is obtained from the absolute humidity and the temperature of the air, for which latter the maximum amount of vapour f' is taken from Table 13, by calculating it as $\frac{f}{f'}$.

(3.) The *tension* e of the water-vapour in the air, which depends on the absolute humidity f, and the temperature t, according to the formula—

$$e = 0{\cdot}943 \, (1 + 0{\cdot}003665t) \, . \, f,$$

$$\text{or } f = 1{\cdot}060 \, . \, \frac{e}{1 + 0{\cdot}003665t};$$

so that the determination of t, and either e or f, suffices for the calculation of all the quantities. (e is measured in millimetres of mercury.)

For the vapour-density of water is 0·623; therefore one cubic centimetre of water-vapour of the tension e, at the temperature t, weighs, since it follows, at ordinary temperatures, Mariotte's and Gay Lussac's law (18), $0{\cdot}623 \cdot \frac{1293}{1 + 0{\cdot}003665t} \cdot \frac{e}{760}$ grm.

I. *Daniell's and Regnault's Hygrometers.*—With these instruments the dew-point—*i.e.* the temperature τ, at which the air is saturated with vapour—is determined directly. In Table 13 we then find for any value of τ between $-10°$ and $+30°$, the corresponding quantity of vapour f in a cubic metre of air, or the density multiplied by 1000000, and also the tension e of the vapour saturated at τ; and this is, without any further calculation, the actual tension in the atmosphere. The density needs a correction, because the air in the neighbourhood of the instrument is cooled, and therefore made denser. The contained water as taken from the table for τ is therefore too great, and must, since the vapour expands practically like a permanent gas, be multiplied by $\frac{1 + 0{\cdot}003665 \cdot \tau}{1 + 0{\cdot}003665 \cdot t} = \frac{273 + \tau}{273 + t}$, where t signifies the temperature of the air.

In Daniell's, as in Regnault's hygrometer, the temperature of the polished surface is made to sink by the evaporation of ether until a dimness from the deposited water is observed. Then the evaporation of the ether is interrupted, the temperature rises, and the reading is taken at which the deposit begins to disappear. After some preliminary trials it is easy to bring the temperatures of the appearance and disappearance of the dew within a small fraction of a degree of each other. The mean of the two is then the dew-point τ of the air. To attain the greatest possible accuracy, Regnault prescribes such a regulation of the flow of water from the aspirator (and therefore of the stream of air through the ether) that the deposit of dew sometimes appears and sometimes disappears. The temperature as read off is then the dew-point without any further trouble. In using Daniell's hygrometer, care should be taken that the moisture arising from

the body, breath, etc., should be kept as far as possible from the surface on which the dew is to be formed.

II. By *Auguste's Psychrometer* [in England usually called Leslie's] the humidity of the air is determined from the rapidity with which water evaporates in the air, which rapidity again, is measured by the cooling of a thermometer the bulb of which is kept wet.

If then

t = the temperature of the air (dry bulb reading);
t' = the wet bulb reading;
e' = the maximum tension of water-vapour at the temperature t', as taken from Table 13;
b = the height of the barometer in mm.;

the actual tension of the vapour e is obtained by the formula—

$$e = e' - 0{\cdot}00074 \,.\, b \,.\, (t - t').$$

When e has been found, the absolute humidity f (water contained in 1 cubic metre of air) may be found by the formula on p. 71.

The constant 0·00074 is for observations in the open air in moderate motion. In still air a larger number must be used; that for a small closed room may be as much as 0·0012. Since general rules for the variation are not known, it is best in observations in a room to fulfil the conditions of the constant 0·00074 by moving the thermometer about.

On account of the many sources of error to which this method of determining e is subject, it is quite sufficient to use for b a mean height of the barometer. If we take $b = 750$, $e = e' - 0{\cdot}56\ (t - t')$. The value of f may be approximately found by the formula $f = f - 0{\cdot}6\ (t - t')$, taking from Table 13 the value of f' corresponding to t'. If psychrometer determinations be made, say in the course of determining a vapour-density, in a moderately large closed balance-room, the tension e will be found with sufficient accuracy as $e = e' - 0{\cdot}75\ (t - t')$.

Example.—It has been found that $t = 19^{\circ}{\cdot}50$, $t' = 13^{\circ}{\cdot}42$, the height of the barometer $b = 739$ mm. We find in Table 13 for t',

$e' = 11{\cdot}44$ mm. From this we must take $0{\cdot}00074 \cdot 739 \cdot 6{\cdot}08 = 3{\cdot}33$ mm.; therefore the tension of the water-vapour $e = 8{\cdot}11$. From this the water contained in 1 cubic metre at the temperature $19°{\cdot}5$ is found, according to p. 71, to be—

$$f = \frac{1{\cdot}060 \,.\, 8{\cdot}11}{1 + 0{\cdot}003665 \,.\, 19{\cdot}5} = 8{\cdot}03 \text{ grm.}$$

By the approximate formulæ—

$$e = 11{\cdot}44 - 0{\cdot}56 \,.\, 6{\cdot}08 = 8{\cdot}04 \text{ mm.; and}$$
$$f = 11{\cdot}59 - 0{\cdot}6 \,.\, 6{\cdot}08 = 7{\cdot}94 \text{ grms.}$$

III. The water contained in 1 cubic metre of air may be obtained directly by drawing a measured quantity of air through a tube filled with calcium chloride or concentrated sulphuric acid, by means of an aspirator, and determining the increase of weight due to the absorption of the water.

29.—Determination of Specific Heat. Method by Mixture.

The specific heat or heat capacity of a body is the ratio of the quantities of heat which must be imparted to the body, and to the same weight of water, to raise the temperature by the same amount in both. If, as is usually the case, we call the quantity of heat which will heat 1 grm. of water 1° the unit, we may say that the specific heat of a substance is the quantity of heat which will raise the temperature of 1 grm. of it by 1°.

Strictly speaking, the specific heat is not a constant quantity, since the amount of heat required for a rise of temperature of 1° increases a little with the temperature. The variation in the case of water is given in Table 14. It is calculated, on the assumption that that quantity of heat is taken as the unit which raises the temperature of 1 grm. of water from 0° to 1°. Usually no account need be taken of this small difference. Where nothing further is said about it, the mean specific heat between 0° and 100° will be taken as that of the substance.

The body to be experimented on (if solid) is weighed,

warmed to a given temperature, put into a weighed quantity of water, and the loss of temperature which it experiences and the rise of temperature of the water determined from the final temperature common to them both.

Then if

T = the temperature of the heated body;
t = the initial temperature of the water;
τ = the common final temperature;
M = the weight of the body;
m = the weight of the water, increased by the equivalent of the rest of the calorimeter (see below);

the specific heat C of the body is found by the formula—

$$C = \frac{m}{M} \cdot \frac{\tau - t}{T - \tau}.$$

It must be noticed that the walls of the vessel and the thermometer in the calorimeter participate in the warming. The vessel is made of thin sheet metal (*e.g.* copper-gilt or thin silver). If γ be the specific heat of the metal employed (Table 14), μ the weight of the vessel, the quantity of heat necessary to heat it from t to τ, will be $\mu\gamma\,(\tau - t)$. The quantity of heat $\mu\gamma$, which raises the temperature 1°, is called *the equivalent in water* of the vessel. The equivalent weight of the thermometer must be determined by experiment. For this purpose it is heated, say by plunging it into heated mercury, about 30°, and then quickly transferred to a weighed quantity of water, and the rise of temperature produced is observed with another delicate thermometer. This multiplied by the mass of the water, divided by the loss of temperature of the heated thermometer, gives its equivalent weight.

For m, therefore, in the formula given above, we must put the sum of the equivalent weights, thus once for all determined, of the solid parts of the calorimeter, added to the net weight of the water used for filling it.

The unavoidable loss of heat from the calorimeter to surrounding objects during the experiment is most easily got

rid of by making the initial temperature, as nearly as possible, as much below the temperature of the room as the final temperature τ will be above it. The rise of the temperature which may be expected is determined by a preliminary experiment, or if the specific heat be approximately known it may be calculated with sufficient accuracy. In order, besides, that it may be sufficient to fulfil this condition approximately, the change of temperature in the calorimeter must not exceed a moderate quantity (10°). Further, to reduce the radiation the vessel should be made of polished metal, and should be placed upon a badly-conducting support (three points of wood or crossed silk threads).

The warming of the body is performed in a space heated from the outside by boiling water or steam from boiling water (in addition to the well-known apparatus of Regnault, see also that of Neumann, *Pogg. Ann.*, vol. cxx., p. 350), and must be continued until a thermometer placed in the enclosure is stationary. Bodies in small pieces must be held together by a light basket of wire-gauze, the equivalent of which is included in the calculations in a manner easily determined. During the observation at the calorimeter the water must be kept well in motion with a little stirrer, the equivalent of which can be determined in the same way as that of the vessel. If water cannot be used, another fluid is taken of known specific heat, Table 14 (*e.g.* oil of turpentine), and the result, calculated as above, must be multiplied by this number.

Example.—1. Equivalent of the vessel and stirrer.—Both parts were made of brass and weighed together, $\mu = 19$ grms. The specific heat of brass is $\gamma = 0{\cdot}094$; the equivalent therefore is $\mu\gamma = 19 \,.\, 0{\cdot}094 = 1{\cdot}8$ grm.

2. Equivalent of the thermometer.—The thermometer was warmed to 45°, and plunged into a small vessel containing 20 grms. of water of the temperature of 16°·25. The temperature then rose to 17°·10. The equivalent of the thermometer therefore amounts to—

$$20 \,.\, \frac{17{\cdot}10 - 16{\cdot}25}{45 - 17{\cdot}1} = 0{\cdot}6 \text{ grm.}$$

The equivalent of the solid parts of the calorimeter is therefore = 2·4 grms.

3. The body weighed $M = 48{\cdot}3$ grms.
The water weighed 74·0 grms.; therefore
$m = 74{\cdot}0 + 2{\cdot}4 = 76{\cdot}4$ „
The temperature of the hot body $T = 96^\circ{\cdot}7$ „
The initial temperature of the water $t = 11^\circ{\cdot}05$ „
The final temperature $\tau = 16^\circ{\cdot}74$ „
(The temperature of the room was 14°.)

Hence we find the specific heat—

$$C = \frac{76{\cdot}4}{48{\cdot}3} \cdot \frac{16{\cdot}74 - 11{\cdot}05}{96{\cdot}7 - 16{\cdot}74} = 0.1125.$$

In order to determine the specific heat of a fluid by the method by mixture, the calorimeter is filled with it, and a weighed body of known specific heat is heated, and the operation conducted as above. If M, T, C be the weight, temperature, and specific heat of the heated body;

t = the initial temperature of the fluid;
τ = the final temperature;
m = the weight of the fluid;
w = the equivalent of the solid parts of the calorimeter;

the specific heat of the fluid is—

$$c = C\frac{M}{m} \cdot \frac{T - \tau}{\tau - t} - \frac{w}{m}.$$

[With small quantities of fluid, the thermometer itself may be used as the hot body, its equivalent in water being known. See *Phil. Trans.*, 1865.]

30.—Specific Heat. Method by Cooling.

Here the times are compared in which heated bodies, which cool under the same conditions, experience the same fall of temperature. The process only furnishes useful results in the case of liquids or powdered solids of good conducting power.

A small vessel of thin polished metal, in which a sensitive thermometer is placed, is filled with the substance. Solid bodies may be tightly jammed down. When com-

pletely full the vessel is closed with a lid. It is then warmed with the substance in it, and introduced into a metal receiver, which can be exhausted by an air-pump, and the temperature and the time are observed. The receiver is kept at a constant temperature by surrounding it with a large quantity of water or with melting ice.

For quantities of fluids not too small the rate of cooling in the air in a single closed metallic vessel may be observed.

Let there be, then, two sets of observations with the vessel filled with two different substances. We will call

m and M the quantities used to fill the vessel;
w the equivalent in water of the vessel and thermometer (p. 75);
ϑ and θ the times during which the bodies cool from the same initial to the same final temperature;
c and C the two specific heats;

then—

$$c = \frac{1}{m}\left[(MC + w)\frac{\vartheta}{\theta} - w\right].$$

For the times necessary to the same amount of cooling are proportional to the quantities of heat given off—*i.e.*

$$\frac{\vartheta}{\theta} = \frac{mc + w}{MC + w}.$$

If, therefore, we know c; *e.g.* by using water $c = 1$, we can from this find c.

If the temperature of the surrounding walls of the receiver be not the same at the two experiments, the temperature of the substances must be taken as the excess over that of the receiver.

Some time must be allowed to elapse after warming the body. It will always be best to make a larger set of observations by noticing the temperature every 30 seconds. Then a curve is constructed from these observations by putting the times as abscissæ, the temperature as ordinates, and from the curves are taken the times which correspond to equal initial and final temperatures (or excess of temperature over that of the surrounding bodies). Thus we can, from one pair of ob-

servations, obtain a large number of determinations, of which the mean is afterwards taken.

Errors of observation have the least influence when the excess of the first temperature over that of the surroundings is about three times that of the second.

31.—Specific Heat. Methods by melting Ice.

The body, of weight m, heated to the temperature $t°$, is placed in dry ice at 0°, and allowed to cool to 0° in it, by giving up its heat to the ice which surrounds it on all sides. If, by this means, the quantity M of ice be melted, the specific heat of the body is—

$$c = \frac{M}{m} \cdot \frac{79 \cdot 4}{t}.$$

The unit-weight of ice at 0° requires 79·4 units of heat to become converted into water at 0°.

The access of heat to the ice calorimeter from the outside is avoided by surrounding it on all sides with melting ice.

In order to determine the quantity of ice melted by weighing, or taking the volume of the water (Lavoisier's and Laplace's ice-calorimeter), with anything approaching to accuracy, we must, on account of the cohesion of the water to the ice, use a large quantity of the body.

Bunsen's Ice-Calorimeter (*Pogg. Ann.*, vol. cxli., p. 1; [*Phil. Mag.*, 1871]).

In this form of instrument the quantity of ice melted is determined by the diminution of volume which is experienced when water passes from the solid to the liquid state. If a mixture of ice and water contract v c.c., whilst a body of the mass of m grms. cools from $t°$ to 0°, the specific heat of the body is—

$$c = \frac{v}{m} \cdot \frac{875 \cdot 4}{t}.$$

1 grm. of ice has, according to Bunsen, the volume 1·09082 cc., whilst 1 grm. of water at 0° has the volume 1·00012. By melting

1 grm. of ice, which requires 79·4 units of heat, there occurs therefore the diminution of volume of 0·0907 c.c. The unit of heat therefore diminishes the volume $\frac{0 \cdot 0907}{79 \cdot 4} = \frac{1}{875 \cdot 4}$ c.c.

Bunsen's calorimeter consists of the parts *a b c*, made of glass, sealed together by the blow-pipe; *d* is an iron piece cemented on; *c* and *d* are filled up to the dotted line with boiled mercury. Above this there is in *b* water freed from air by boiling, in which the ice is formed as a coating to *a* by passing through *a* a stream of alcohol which has previously been cooled in a freezing mixture.

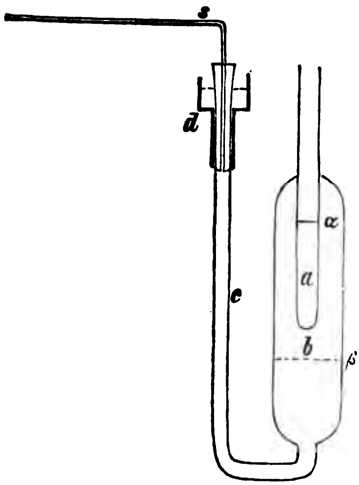


Fig. 4.

When in use the instrument is fixed by *d* in a holder, surrounded with melting snow, and the calibrated scale-tube *s* pressed through a long cork fixed in *d* until the mercury stands sufficiently far along the divisions. When the vessel *a* has been filled up to *α* with water or some other fluid which does not dissolve the body to be experimented on, this latter is heated, and let fall into *a*, which is then closed by a cork. The mercury in *s* sinks, and finally becomes stationary. If the movement of the mercury amount to *p* scale-divisions, and if the value of 1 division be φ, $v = p\varphi$.

We get φ by determining the weight, μ grammes, of a thread of mercury which occupies *n* divisions. If τ be the temperature at the time when this measurement is made—

$$= \frac{\mu\,(1 + 0 \cdot 00018\,\tau)}{13 \cdot 596 n} \text{ c.c.}$$

32.—Comparison of the Conducting Power for Heat of two Rods.

We assume that the two rods have the same section, and we give them a similar condition of surface by covering them with some opaque varnish, or by polishing and electroplating with silver. The two ends of the rod are brought to different temperatures, say by surrounding one end with boiling water and the other with melting ice. An inferior method is to leave one end exposed to the air, and heat the other by a lamp which burns very regularly. The middle part of the rod, at which the following determinations of temperature are made, is protected by screens from the radiation of the source of heat. The distribution of temperature, after a time, becomes constant.

When this state has been arrived at, the temperatures of three points of the rod equally distant from each other, I. II. III. are measured. The excess of temperature over that of the surrounding air may be called t_1, t_2, t_3.

Let us call

$$\frac{t_1 + t_3}{t_2} = n.$$

The same course of proceeding is now gone through with the other rod. The excess of temperature at the three points at the same distance from each other as before we call T_1, T_2, T_3, and also—

$$\frac{T_1 + T_3}{T_2} = N.$$

Then the two conductivities k and K are in the ratio—

$$\frac{K}{k} = \left[\frac{\log\left(\frac{n}{2} + \sqrt{\frac{n^2}{4} - 1}\right)}{\log\left(\frac{N}{2} + \sqrt{\frac{N^2}{4} - 1}\right)}\right]$$

The temperature is determined by plunging small delicate thermometers into holes, as small as possible, bored in the rod, and filled with oil, mercury, or fusible metal. The thermo-

meters must be accurately compared with each other. It is better to use a thermo-element (25).

Proof.—When the thermal condition of the bar has become stationary, each unit of length of the rod receives by conduction the same amount of heat as is given off by it to the surrounding air. Let t be the excess of temperature above the surroundings. The quantity of heat given off in each unit of time is therefore at, where a is the same for both rods. The quantity of heat carried by conduction is $k.q. \frac{d^2t}{dx^2}$, if x denote the distance from the end surface, k the conducting power, and q the section, the same for both rods. If we put $\frac{a}{kq} = \alpha^2$, α^2 is a quantity inversely proportional to the conducting power under consideration, and the differential equation for the stationary condition as to temperature will be—

$$\frac{d^2t}{dx^2} = \alpha^2 t.$$

The complete integral of this equation is—

$$t = Ce^{\alpha x} + C'e^{-\alpha x},$$

where C and C' are two constants depending upon the heating of the end surfaces. If we call t_1, t_2, t_3 the temperatures of three sections lying at the distance l from each other, we easily find, by putting x_1, $x_1 + l$, $x_1 + 2l$, in the above equations, after eliminating C and C', the expression—

$$e^{\alpha l} + e^{-\alpha l} = \frac{t_1 + t_3}{t_2} = n \text{ (see above), or}$$

$$e^{\alpha l} = \frac{n}{2} + \sqrt{\frac{n^2}{4} - 1}.$$

We have therefore for equal values of l, α proportional to the expression $log \left[\frac{n}{2} + \sqrt{\frac{n^2}{4} - 1}\right]$, or the conducting power to the reciprocal of the square root of this magnitude.

33.—Determination of the Modulus of Elasticity of a Wire or Rod by Stretching.

The upper end is fastened to the wall or to some solid support, and the lower end loaded if necessary with a weight sufficient to keep it stretched; its length is then measured. An additional weight is now put on to the lower end, and the increase of length which it causes is measured. Calling

L, the length ;
P, the additional load ;
l, the increase of length caused by it ;
Q, the sectional area of the wire ;

the modulus of elasticity E of the stretching is

$$E = \frac{L}{l} \frac{P}{Q}.$$

The original length and the stretching are both to be measured in the same unit. The amount of the number E is of course dependent upon the units in which section and weight are measured. The square millimetre and kilogramme are usually employed, and the number so obtained may be distinguished by the affix $\frac{\text{kgr.}}{\square\ \text{mm.}}$.

If the upper end of the rod be immovable, as is usually the case, the stretching may be measured by the motion of a mark on the lower end. Commonly, however, it is better to make a mark at the upper and lower ends, and to measure the distance between them with each load.

For measurements with a microscope movable on a divided rod, or a cathetometer, the marks are made as two fine cross-strokes with a diamond or fine file.

The amount of elongation employed in the measurement must always be within the limits of elasticity; that is to say, the wire must, on the removal of the weight, return to its original length, a condition the fulfilment of which should be verified by experiment. The limit of elasticity may be widened by loading the wire heavily before the experiment. Even with hard metals the weight employed in the measure-

ments should not exceed half the breaking-strain. (See Table 17 for the tensile strength of some substances.)

The accuracy of the method will be considerably increased if the length be observed under many loads. (See the example.)

The area of the section may be obtained by measurement of the diameter, for which, when small, a micrometer or microscope (45) must be employed.

But the area may also be found by weighing a measured length. The specific gravity Δ (Table 1) of the substance being known, if we find the weight, m mgr., of h mm. of the wire, the section $Q = \frac{m}{h \cdot \Delta}$ $\square$ mm.

Example of a determination of the modulus of elasticity of an iron wire.—Two metres of the wire weighed 1210 mgr. The sp. gr. was found = 7·575, whence it follows that the section

$$Q = \frac{1310}{2000 \cdot 7{\cdot}575} = 0{\cdot}08647 \ \square \ \text{mm.}$$

By Table 17 the tensile strength of this iron wire = 0·08647 × 61 = 5·4 kgr. The heaviest load used in the experiment ought therefore to be 2·7 kgr.

A weight of about ½ kgr. was necessary to straighten the wire, and is not included in the following numbers. Before the observations, the wire was for some time loaded with a weight of 4 kgr.

The following observations of the distance between two points were made in order with different loads:—

No.	Load. kgr.	Length. mm.	No.	Load. kgr.	Length. mm.	Elongation. for 2 kgr. mm.
1	0·0	913·80	2	2·0	914·91	1·11
3	0·1	913·86	4	2·1	914·85	1·09
5	0·2	913·90	6	2·2	915·00	1·10
7	0·3	913·98	8	2·3	915·09	1·11
						Mean = 1·102

The elongation hence is l = 1·102 mm.
For an increase of load P = 2 kgr.
Hence the modulus of elasticity is (*vide supra*)—

$$E = \frac{L}{l}\frac{P}{Q} = \frac{913{\cdot}8 \times 2}{1{\cdot}102 \times 0{\cdot}08647} = 19180 \ \frac{\text{kgr.}}{\square \ \text{mm.}}$$

34.—MODULUS OF ELASTICITY FROM LONGITUDINAL VIBRATIONS.

A rod or wire, the latter stretched and fixed at both ends, is rubbed longitudinally, and so caused to sound. By comparison with a tuning-fork of known pitch, the time of vibration is determined. Calling

L the length of wire, in metres;
Δ its specific gravity;
9810 the acceleration by gravity in millimetres;
n the number per second of the longitudinal vibrations (Table 18);

the velocity of sound in the substance is $c = 2nL$, and the modulus of elasticity

$$E = \frac{4n^2 L^2 \Delta}{9810} \frac{\text{Kgr.}}{\square \text{ mm.}}.$$

Proof.—We have $c = \sqrt{\frac{Eg}{\Delta}}$ where g = the acceleration by gravity, and Δ the weight of unit volume of the substance. Since we have chosen the mm. as the unit of length, and the kgr. as unit of weight (which is not systematic since mm. corresponds to mgr.), we must put for Δ the weight of a cubic mm. in kgr. But it obviously comes to the same thing to take Δ as the specific gravity, and express L in metres but g in mm.

The longitudinal vibrations are produced by friction with a woollen cloth, which, for wood or metals, is sprinkled with resin, for glass is damped. A wire stretched, and fastened at both ends, is rubbed in the middle, a rod is clamped by the middle part, and rubbed on one half.

With a stretched wire, which can be lengthened or shortened, the observations are made more exactly by bringing its note into unison with the fork, than by estimating the interval between the notes. When the notes produced are very high, it is often difficult to distinguish between them and their octaves. Such an error, will, however, be easily detected in the results, as it will make them at least four times

too much or too little, and the true value is usually already known within narrow limits.

The modulus of elasticity deduced from longitudinal vibrations is usually some 1 or 2 per cent higher than that determined from stretching, since between loading the wire and measuring the length time elapses, during which a slight elongation is inevitable by virtue of the elastic action.

Example.—The above-mentioned iron wire of the length of 1·361 metre, gave the note $A\sharp_3$, which is found by Table 18 to be produced by 1865 vibrations per second. The specific gravity is 7·575; therefore

$$E = \frac{4 \times 1865^2 \times 1{\cdot}361^2 \times 7{\cdot}575}{9810} = 19{,}900 \ \frac{\text{Kgr.}}{\square \text{ mm.}}.$$

Another Definition of the Modulus of Elasticity.—The use of the above formula assumes the modulus of elasticity of a body to be that weight (kgr.) which must be hung on a wire of 1 □ mm. section to double its length, supposing that the elongation is proportional to the load.

Another definition, which in practice is preferred, takes, instead of the section, the weight of the unit of length, and considers the modulus of elasticity as the load which would double the length of a wire of which unit length has the unit weight (*e.g.* of which 1 mm. weighs 1 mgr.) We may also define it by supposing the weight necessary to double the length of the wire to consist of a similar wire. The modulus of elasticity would be the length of this wire, which, to correspond to the above definitions, must be measured in kilometres.

The modulus of elasticity E' of the last two definitions may be obtained from E by dividing by the density of the substance. I. In the measurement by stretching, retaining the notation of p. 83 for L, P, and l, but taking in addition m as the mass of unit length (mgr. and mm.)—

$$E' = \frac{L}{l} \cdot \frac{P}{m} \text{ kilometres.}$$

II. In the measurement by the musical note (*vide supra*)—

$$E' = \frac{4n^2L^2}{9810}.$$

In I., instead of a measurement of the section, a simple weigh-

ing of a measured length is all that is required, and II. is independent of any weighing at all.

From the example, p. 84, the numerical value of the modulus of elasticity, according to the last definition, will be for an iron wire, of which 1 mm. weighed 0·655 mgr.—

$$E' = \frac{913{\cdot}8 \times 2}{1{\cdot}102 \times 0{\cdot}655} = 2532 \text{ kilometres};$$

and from the example, p. 86—

$$E' = \frac{4 \times 1865^2 \times 1361^2}{9810} = 2670 \text{ kilometres.}$$

35.—Modulus of Elasticity by bending a Rod.

I. A horizontal rod is clamped tightly at one end, and the position of the free end observed on a vertical scale (*e.g.* a cathetometer). It is then loaded with a weight of P kilogrammes on the free end, and the amount of deflection s thus produced is observed. Let the free length of the rod be l. Then the modulus of elasticity E, if the section of the rod be a rectangle with the vertical side a and the horizontal b, is

$$E = 4\,\frac{P}{s}\cdot\frac{l^3}{a^3 b};$$

if the section be a circle of radius r—

$$E = \frac{4}{3}\cdot\frac{P}{s}\cdot\frac{l^3}{r^4\pi}.$$

II. The difficulty of getting a perfectly tight clamping is avoided by laying the rod with both ends loose upon two solid supports. Let the distance of the two supports from each other be l. A weight P is then hung from the middle of the rod and produces the deflection s', and we have, for rectangular section (*vide supra*)

$$E = \frac{1}{4}\,\frac{P}{s'}\,\frac{l^3}{a^3 b};$$

for circular section

$$E = \frac{1}{12}\,\frac{P}{s'}\,\frac{l^3}{r^4\pi}.$$

P is expressed in kgr. all lengths in mm., in order to get our result in the ordinary unit of the modulus of elasticity (p. 83).

The formulæ given above assume that the deflections are small compared with the length. We must also make sure that the change of form is within the limit of recovery—*i.e.* that on taking away the weight the original form is resumed. Small sections are determined by weighing (p. 84), and the above formulæ may then be simplified by reflecting that ab and $r^2\pi$ are the respective sections of the rods.

The equation under I. for rectangular rods is got thus:—When the rod is bent the fibres at the top are stretched, those at the bottom compressed, the middle layer remains of unaltered length. We denote by x the horizontal co-ordinate of a point of this neutral plane measured from the fixed point, and by y the vertical co-ordinate; and then the curvature of the rod at any point will be $\frac{d^2y}{dx^2}$, for we assume that the bending is small. If now z be the distance of a fibre from the neutral plane (above being reckoned positive, below negative), a small portion of the fibre is stretched or compressed in the ratio $z\frac{d^2y}{dx^2}$ to its original length. A lamina of the breadth b, and thickness dz, seeks therefore to draw itself together with the force $Ez\frac{d^2y}{dx^2}bdz$, and these forces in the laminæ, distant $+z$ and $-z$, produce a couple equal to $2Ez\frac{d^2y}{dx^2}bdz$. The couple, therefore, developed in one entire section of height a and breadth b, is—

$$2Eb\frac{d^2y}{dx^2}\int_0^{\frac{a}{2}} z^2dz = Eb\,\frac{a^3}{12}\,\frac{d^2y}{dx^2}.$$

This couple, produced by the elasticity, must be equal to the statical moment $P\,(l-x)$, exercised by the weight at the place, therefore—

$$\frac{d^2y}{dx^2} = \frac{12}{E}\,\frac{P}{a^3b}\,(l-x).$$

By double integration we get the deflection at the point x—

$$y = \frac{12}{E}\cdot\frac{P}{a^3b}\cdot\left(\frac{lx^2}{2}-\frac{x^3}{6}\right);$$

therefore the deflection of the end, where $x = l$—

$$s = \frac{4}{E} \cdot \frac{Pl^3}{a^3 b}.$$

Further, that for the same rod, if both ends be left loose, the weight in the middle produces $\frac{1}{16}$ of the deflection, follows from the fact that we may in this case consider the middle fixed, and each end drawn up with a weight of $\frac{P}{2}$.

36.—Modulus of Elasticity of a Wire by Swinging under Torsion.

A weight is hung on the wire and is set swinging round a vertical axis passing through its point of suspension. Calling

l the length of the wire in mm.;
r its radius in mm.;
K the moment of inertia of the swinging weight in Kgr., $\square$ mm., taken round the axis of revolution (53);
t the time in seconds of an oscillation (51);

we have the modulus of torsion for the substance of which the wire is made—

$$T = \frac{Kl\pi}{9810 t^2 r^4} = 0{\cdot}000320 \frac{Kl}{t^2 r^4}.$$

If a cylinder, with its axis vertical, be used as the weight, we must put $K = \frac{MR^2}{2}$, where R is the radius in mm., M the mass in kgr.

The modulus of torsion T is about a fifth part of the modulus of elasticity E (33), but the ratio may vary beween $\frac{1}{4}$ and $\frac{1}{6}$.

With the stretching of a rod by hanging on a weight there is always experienced a diminution of diameter. If l be the length, d the diameter, δ the diminution of this latter, which is connected with the stretching λ, and we call the ratio of the diminution of section to the stretching in length $\frac{\delta}{d} : \frac{\lambda}{l} = \mu$, we have by the theory of elasticity $T = \frac{1}{2} \frac{E}{1 + \mu}$. It is found by experience that $\mu \begin{smallmatrix} >0 \\ <\frac{1}{2} \end{smallmatrix}$, and

therefore in every case $T \begin{smallmatrix} <\frac{1}{4}E \\ >\frac{1}{6}E \end{smallmatrix}$. For the mean value $\mu = \frac{1}{4}$, T would be $\frac{1}{5}E$.—(*Poisson.*)

37.—Determination of the Velocity of Sound by Dust Figures (*Kundt*).

The velocity of sound in dry atmospheric air at 0° is $330 \frac{\text{metre}}{\text{second}}$, but in dry air at temperature t, $330\sqrt{1 + 0{\cdot}003665t}$. In the ordinary state of the air as to humidity we can at moderate temperatures consider approximately $c = 330\sqrt{1 + 0{\cdot}004t}$ (see 18).

This number may be used to determine the pitch (number of vibrations) of a rod or tube rubbed longitudinally. The rod is laid horizontal and fixed in the middle. One end, E, is

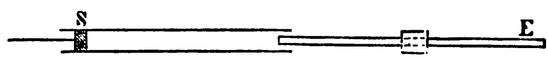


Fig. 5.

rubbed longitudinally, the other is inserted into a glass tube, at least 20 mm. wide, closed at the other end by a movable plug. The tube must be well cleaned, and covered on the inside with lycopodium or sifted cork-dust. On rubbing the rod the impulses of the free end produce stationary air-waves in the glass tube, by which the powder is arranged in regular figures. By altering the position of s the place is easily found at which the agitation of the powder is most energetic, and at this the plug is left. A light cork or card-board disc may be fixed to the end of a rod of small section in order to facilitate the communication of the impulses to the column of air.

Afterwards the length l of the waves of dust is measured by laying a divided scale underneath, and if L be the length of the rod which is rubbed, the velocity of sound in this latter is

$$C = 330\sqrt{1 + 0{\cdot}004t}\,.\,\frac{L}{l}\ \text{metre};$$

and therefore the modulus of elasticity—

$$E = \frac{C^2\Delta}{9810}\ \frac{\text{kgr.}}{\square\ \text{mm.}}\ \text{(p. 85)},$$

where Δ denotes the density of the rod.

As will readily be seen, the experiment may be used to compare the velocity of sound in other gases with that in air.

To obtain the length of the waves as accurately as possible, the distance of two nodes, distant several (n) wave-lengths from each other, is measured and divided by n.

If we wish to use all the waves, we read the positions p_0, p_1, p_2 ..... p_n, of all the nodes, on the scale. The method of least squares shows (3) that the most likely result for the wave-length is obtained by the formula—

$$l = 6\,\frac{n\,(p_n - p_0) + (n-2)\,(p_{n-1} - p_1) + (n-4)\,(p_{n-2} - p_2) + \dots}{n\,.\,(n-1)\,(n-2)}.$$

Example.—A glass rod 900 mm. long gave, at the temperature 17°, nodal points at the divisions 25, 88, 152, 213, 277 mm.; hence—

$$l = 6\,\frac{4\,(275 - 25) + 2\,(213 - 88)}{4\,.\,5\,.\,6} = 62{\cdot}9 \text{ mm.}$$

The velocity of sound in the glass was therefore

$$330\,\sqrt{1 + 0{\cdot}004\,.\,17}\,.\,\frac{900}{62{\cdot}9} = 4890 \text{ metres;}$$

and the modulus of elasticity of the glass—

$$E = \frac{4890^2\,.\,2{\cdot}7}{9810} = 6580\,\frac{\text{kgr.}}{\square\text{ mm.}}.$$

38.—Measurement of an Angle of a Crystal by Wollaston's Reflecting Goniometer.

The instrument is so placed that its axis is parallel to a distant horizontal mark, O, such as a window-frame or roof-ridge, perpendicular to the line of sight. We will first assume that the crystal has been already fixed to the axis, according to the instructions given on the next page, so that the edge of the crystal over which the angle is to be measured lies in the axis, and is parallel to it. Holding, now, the eye close to the crystal, the observer turns the axis until the image of

the upper mark, as seen in one of the crystal-faces, coincides with a lower horizontal mark, U, seen directly. The edge of the floor or the reflected image of the upper mark, as seen in a properly inclined mirror fastened behind the goniometer, may be used for this purpose. The position of the index (vernier) is then read off on the graduation of the circle. Then the circle with the crystal is turned until the image of O, as reflected in the other face of the crystal, coincides with U, and the index again observed. The angle through which the circle has been turned is the supplement of the required angle.

For accuracy in the measurement of the angle there is usually, inside the axis on which the divided circle turns, a second axis concentric with the first, which is used to *repeat* the measurement in the following manner:—When the two processes mentioned above have been gone through, the *first* surface of the crystal is brought into position again by means of the inner axis, *without altering the position of the circle;* then by turning the outer axis with the circle, the *second* face; and the same process is repeated. If, now, n turnings of the circle have been made, the total angle through which the circle has been turned, divided by n, is the supplement of the crystal angle.

To obtain the total angle we take the difference between the first and last reading, and add, according as the circle is graduated to 180° or 360°, twice or four times as many right-angles as the number of times that the zero has passed the index. It is therefore only necessary to read off the first and last positions of the circle.

If the intermediate positions of the circle have also been read off, we may, in order to make use of all the observations, calculate in exactly the same way as in the determination of the length of the waves at the end of the previous article.

Exactly the same method is also used in measuring an angle with a repeating theodolite.

More minute instructions as to the adjustment of the crystal must now be given. Two axes of rotation, perpendicular to each other, would be sufficient to give the edge to

be measured any position (Wollaston's original arrangement). But in this case the desired adjustment can only be attained by trial. If, however, a third axis of rotation be added, the edge to be measured may be made parallel in a regular manner.—(*Naumann.*)

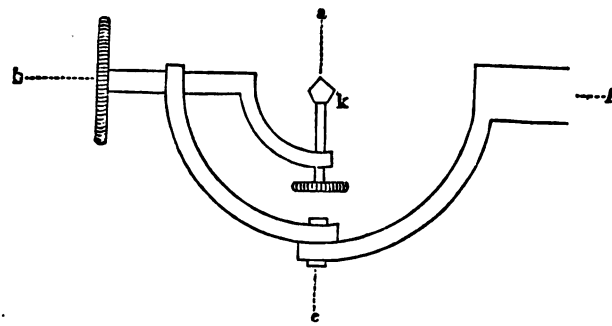


Fig. 6.

A is the axis of the circle; *a b c* the axes for the adjustments; *k* the crystal held upon a piece of wax.

(1.) By turning round *c* a position is found in which *b* is in *A* produced, *i.e.* in which turning *A* does not disturb the milled head *b* from its place; then by turning *a* the face (1) of the crystal is placed parallel to *A*. (See below.)

(2.) The axis *c* is turned through an angle of some 60° or 90°; the face (1) will usually be found to have altered its position with regard to the axis *A*. By turning *b* it is again placed parallel to *A*. The face (1) is by this means made parallel to *A* and *b*, and therefore perpendicular to *c*; no turning of *c* will there affect the position of (1).

(3.) By turning *c* the face (2) is made parallel to *A*.

In each successive adjusting of an axis, those already brought into position must not be altered.

In order to tell whether a face is parallel to the axis *A*, the two points in the upper and lower marks are noted, one of which is perpendicularly under the other in the plane of the divided circle. If one of the horizontal window-bars have been used as a mark, it will be most convenient to make use of one of the vertical bars, and for the lower point

that at which a plumb-line hanging from it cuts the lower mark. With the roof-ridge a chimney is chosen, or a lightning-conductor, and underneath its image in the fixed mirror. Of course the goniometer is always placed in the plane passing through the vertical marks, which is perpendicular to the horizontal ones. The face of the crystal is parallel to the axis, as soon as by a suitable rotation round A the image of the upper point in the face is made to coincide with the lower one.

Before accurately adjusting the crystal, it should be seen by a rough trial that in the position of the circle necessary for an observation one of the arms (see Fig. 6) would not come between the eye and the lower mark.

39.—Determination of a Refractive Index with the Spectrometer.

The body of which the index of refraction is to be measured, is given as a prism, which is got in the case of a solid by grinding, in the case of a liquid by pouring it into a hollow prism with sides of glass, the surfaces of which should be parallel to each other. The problem divides itself into two parts: the measurement of the angle of the prism, and the deflection of the ray of light.

I. *Measurement of the Angle of the Prism.*

(*a*) When the telescope of the spectrometer is fixed, and the circle is movable. The prism is so placed that the refracting surfaces are equidistant from the axis of the circle, *i.e.* so that by moving this latter one of the surfaces takes the former place of the other. By means of the footscrew of the levelling stand upon which it is placed, the prism is adjusted with its refracting edge parallel to the axis of rotation of the circle. To accomplish this the levelling screws must be altered till a distant point, or a mark on the slit, occupies the same position on the cross-wires of the observing telescope, when reflected in either face of the prism; then by turning the circle the image of some distant vertical object,

or of the slit attached to the spectrometer, reflected from one face of the prism, is made to coincide with the cross-wires, and the position of the circle is read with the vernier. The same is repeated with the other face; the difference of the two readings subtracted from 180° gives the required angle of the prism φ. If the cross-wires of the telescope can be illuminated they themselves are most simply used as the reflected object.

(*b*) When the circle is fixed and the telescope movable with the vernier. The prism is placed with its refracting edge towards the slit, so that the line bisecting it would approximately pass through the slit or some distant object. The cross-wires of the telescope are made to coincide with the image of the object or slit reflected from the two prism-faces successively, and the difference of the readings is double the refracting angle.

The object must be at such a distance that the size of the prism is of no account in comparison with it. If the slit be used its tube must be so drawn out that the rays from it passing through the lens fall parallel on the prism, so that it may appear as an infinitely distant object. To accomplish this, the telescope should first be focussed for parallel rays, so that the image of a very distant object falls in the same plane as the cross-wires. This condition is fulfilled when, on moving the eye sideways before the eye-piece, the image and cross-wires show no parallax, but remain in coincidence. The telescope is then pointed to the slit, the tube of which is drawn out till the image shows no parallax with the cross-wires. It then appears as an infinitely distant object.

II. *Measurement of the Angle of Deviation.*

The direct adjustment of the telescope upon the slit gives the direction of the unrefracted ray. There are two methods by which the deviation of a ray which has passed through the prism, and from this the refractive index, may be found.

(*a*) Usually the prism is placed in the position of *minimum deviation.* The slit is observed through the telescope and prism, and the latter turned until the image of the slit

moves to the same side, whether the prism be turned to the right or left. Here the prism has the position of minimum deviation, and it is fixed and the circle read off when the cross-wires and the image of the slit have been made to coincide. The difference between this position and the direct adjustment gives the angle of deviation δ. The refractive index μ, is then, if we call the refracting angle φ, calculated by the formula—

$$\mu = \frac{\sin \frac{1}{2}(\delta + \varphi)}{\sin \frac{1}{2}\varphi}.$$

Proof.—When a ray of light passes from a vacuum into any body, the angles which it makes with the normal to the refracting surface before and after passing through this are called the angles of incidence and refraction respectively. The ratio of the sines of the two angles is constant, and is called the refractive index or index of refraction of the body. The minimum deviation of a ray passing through a prism is produced when the ray makes, within the prism, equal angles with the two refracting faces, and therefore also with the two normals. These latter angles are $\frac{1}{2}\varphi$ (see Fig. 7). Let the angle of incidence, and therefore of emergence, from the prism be $= \alpha$, therefore $\sin \alpha = \mu \sin \frac{\varphi}{2}$. The angle of deviation of the ray is $\delta = 2\alpha - \varphi$, therefore $\sin \frac{1}{2}(\delta + \varphi) = \sin \alpha = \mu \sin \frac{\varphi}{2}$, from which the formula given above follows.

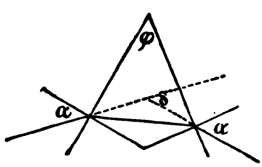


Fig. 7.

(*b*) The prism is placed with the face which is turned towards the telescope perpendicular to it, *i.e.* so that the reflected image of the cross-wires coincides with the same as seen directly. The method assumes that the cross-wires can be illuminated. If we have, again, the angle of deviation δ, the refracting angle of the prism φ,

$$\mu = \frac{\sin(\delta + \varphi)}{\sin \varphi}.$$

Proof similar to that given above.

The index of refraction must of course refer to light of one particular colour. In sunlight, which is thrown horizontally upon the slit by a heliostat, Fraunhofer's lines may be employed, the most characteristic of which are shown in the accompanying figure in their relative positions, as seen

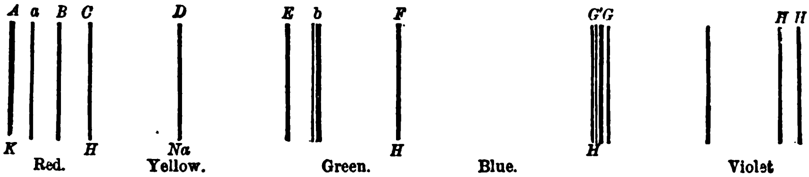


Fig. 8.

through a flint-glass prism. In order to see *A* and *a* the slit must not be too narrow, and a red glass should be held before it. With a narrow slit and greater magnifying power, *D* is seen to be a very close double line.

Where sunlight cannot be used, the line *A* may be obtained by means of the potassium flame, *D* by the sodium flame, *C*, *F*, and *G′* by the light of the electric spark in a narrow Geissler's tube filled with rarefied hydrogen.

The difference of the refractive indices for *A* and *H* (Fraunhofer) is called the dispersive power or dispersion. As mean refractive index that for about *E* is usually taken.

To reduce indices of refraction, measured in the air to their equivalents *in vacuo*, they must be multiplied by 1·00029, which is the index of refraction for light passing from a vacuum into air.

40.—Spectrum Analysis.

The apparatus for spectrum analysis requires, besides the telescope and slit previously mentioned as forming the spectrometer, a third tube with a micrometer scale. This is reflected in the face of the prism which is next the telescope.

The adjustment of the spectrum apparatus is accomplished in the following manner—the *order* of the operations being specially observed :—

(1.) The slit must appear as a very distant object. If the right adjustment be not indicated by a mark on the tube, the telescope must be focussed on some distant object, pointed to the slit, and the latter drawn out till it appears clear and sharp.

(2.) The prism must be adjusted to the position of minimum deviation. To attain this end, where the prism has not been fixed in the proper position by the maker, the slit is illuminated with the sodium flame, and the prism placed approximately in its right position before the collimating lens; and when the direction of the refracted ray has been found with the naked eye, the image of the slit is sought with the telescope. The prism is then turned (following the image, if necessary, with the telescope), until the image stops and begins to move backwards, and is then fixed in this position.

(3.) The reflected image of the scale should be clearly visible. It is illuminated by a lamp placed about 20 cm. from it. When, by turning the tube, the image is found, the tube is drawn out till the scale appears distinct. The images of the slit and scale should not alter their relative positions in the telescope on moving the eye before the eye-piece.

(4.) The sodium line should be made to fall upon some particular division of the scale—that adopted by Bunsen and Kirchhoff being the 50th. This adjustment is made by turning the tube carrying the slit, which should then be clamped.

In order to know the points of the scale which correspond to the lines of the different chemical elements, their flames should be observed separately, and the position on the scale (with their brightness, width, colour, and sharpness) of the lines should be noted. It is more convenient to employ for this purpose the copies which are published of Bunsen and Kirchhoff's maps, or Table 19, which was drawn up by means of Bunsen's apparatus. For this purpose the scale of the apparatus may be reduced to that of the charts in the following manner:—

The positions of a few known lines near the ends and in the middle of the spectrum (in sunlight say *a, D, F, G, H*, or $K\alpha$, $Na\ \alpha$, $Sr\ \delta$, $K\beta$) are observed on the scale of the instrument. The observed positions are laid down on square-ruled paper as abscissæ, and the corresponding positions on Bunsen's scale as ordinates, and a curve drawn through the points obtained. This will seldom differ much from a straight line. On this the position on Bunsen's scale, corresponding to any observed position on that of the instrument, will be found as the ordinate. In many spectroscopes the scale is made nearly to agree with Bunsen's. When this is the case, $Na\ \alpha$ is made to coincide with the 50th division; the scales are compared by a series of observations. The curve is more conveniently constructed, only for the corrections of the scale, by taking the differences from Bunsen's scale as ordinates in the graphical construction. (See Table 19.)

The following remarks must also be attended to:—First, not only must the positions of the observed lines be noted, but, approximately at least, their brightness, width, and sharpness. For instance, $Sr\ \beta$ and $Li\ \alpha$ fall very near together; but, while $Sr\ \beta$ is hazy, $Li\ \alpha$ is quite sharp. For distinguishing the alkaline earths, it is best to make use of the faint characteristic lines in the blue part of the spectrum ($Sr\ \delta$ and *Ca*).

The bodies are always placed in the front border of the flame on platinum wire, the glowing, solid part, so far down that it does not give any disturbing continuous spectrum. It is advisable to use, first of all, a narrow slit, in order to separate lines lying close together, and then to repeat the observation with a wider one, to detect lines of feeble brightness. Similarly, it is well to employ first, a small gas-flame for easily vaporisable bodies (*K, Li*), and then a larger one for those which are less so (*Sr, Ba, Ca*). The spectra of the latter often require a longer time before they make their appearance.

The bodies are usually employed as chlorides; sodium, however, on account of the decrepitation of common salt, is more conveniently used as carbonate. The enfeeblement of

the spectrum, in the course of a long experiment, is often due to the fact that the chlorides by ignition are converted into the less volatile oxide. The intensity of the light is in this case momentarily restored by moistening the bead with hydrochloric acid. The most effectual way of cleaning a platinum wire from a substance volatile with difficulty, is moistening it with hydrochloric acid and long ignition in the point of the blowpipe-flame.

Extraneous light must be carefully cut off; by a black screen behind the lamp, by a cover for the prism, which only leaves a passage for the light through the three tubes, and, lastly, by a black paper shade hung from the telescope. The last renders the wearisome closing of the eye not in use unnecessary. The scale should never be more strongly illuminated than is necessary for recognising the divisions and numbers. In order to see very faint lines, the light passing through the scale may be entirely cut off.

41.—Measurement of the Wave-Length of a Ray of Light.

This measurement is made most simply and accurately with the spectrometer (39), upon the table of which is placed, instead of the prism, a plate of glass with a very fine grating of lines (Nobert's test-lines); the lines parallel to the slit, the plate perpendicular to the tube carrying the slit, the engraved face turned towards the telescope. Using, then, homogeneous light, we shall observe, in suitable positions of the telescope, besides the middle bright image of the slit, two or more fainter images on each side of the middle. Let l be the distance between the lines in the grating on the glass plate, δ_1, δ_2, δ_3 ... the angles of deviation of the images from the middle one, the wave-length of the light used is—

$$\lambda = l \sin \delta_1 = \tfrac{1}{2}\, l \sin \delta_2 = \tfrac{1}{3}\, l \sin \delta_3, \text{ etc.}$$

We can render ourselves independent of the accurately-perpendicular position of the grating, by measuring not the distance of the image from the middle one, but the distance

between two corresponding images on opposite sides, and dividing by two.

Light not homogeneous is dispersed by the grating into spectra, in which, according to the formulæ given above, the light consisting of the longer waves (red) appears most deflected. In using sunlight in which the Fraunhofer's lines (p. 97) are used for the definition and adjustment of the colour, the first spectrum and the greater part of the second are pure; beyond this the spectra are superimposed. In order to recognise the lines in interference-spectra from a map of the dispersion-spectrum (p. 97), it must be remembered that the former appears more and more contracted the more the violet end is approached.

42.—Measurement of a Radius of Curvature with the Spherometer.

The radius of curvature of a spherical surface—*e.g.* the surface of a lens—can, when it is large enough, be determined with the spherometer in the following manner:—

The instrument must first of all be placed upon a surface known to be plane. By turning the micrometer screw, the middle foot of the spherometer is made of such a height that all four points rest on the surface. This adjustment is known with great exactness from the fact that, with a slightly lower placing of the middle point, the instrument rocks, and can be easily turned round the point.

The instrument is then placed upon the surface, the radius of curvature of which is to be determined, and the screw is again turned until all the points rest on the surface at the same time.

The positions of the middle point in the two experiments differ by a certain number of revolutions of the screw. The whole numbers will be reckoned by the number of turns of the screw, or by the scale at the side, of which each division corresponds to one turn of the screw; the fractional parts on the divided circle on its head. The number of revolutions, multiplied by the distance between two threads (in

millimetres), gives the height of the middle point above the plane of the three outer fixed ones, when all four rest on the curved surface. Let

a = this distance;
l = the side of the equilateral triangle formed by the three fixed feet as angles;

then the required radius of curvature r is—

$$r = \frac{l^2}{6a} + \frac{a}{2}.$$

Example.—From the position in which all four points rest on a plane, the centre point had to be raised 6·272 turns of the screw in order that all four might rest on the surface of a convex lens. The distance between the threads of the screw = 0·5 mm., hence $a = 3{\cdot}136$ mm. The side of the equilateral triangle formed by the three fixed points is $l = 82{\cdot}5$ mm. Hence the radius of curvature of the surface of the lens is—

$$r = \frac{82{\cdot}5}{6\,.\,3{\cdot}136} + \frac{3{\cdot}136}{2} = 363{\cdot}3 \text{ mm.}$$

The way in which the spherometer may be employed to ascertain the thickness of a plate, or the parallelism of the surfaces of a plate, or the spherical form of a surface may be tested by it, is clear without further explanation.

43.—Radius of Curvature by Reflection.

The determination of the radius of curvature by the spherometer is limited to large surfaces. In order to determine that of a small surface, if reflecting, we may proceed as follows:—

The object is arranged so that the surface to be measured is perpendicular, and two lights are placed in front of it at the same height as the object, and at the same distance from it. Half-way between the lights is placed a telescope directed towards the surface. Lastly, immediately in front of the surface is placed a small scale divided on glass. The lights produce two images reflected from the surface, of

which the distance apart is observed on the small scale with the telescope. If then

l = the distance of the images from each other;
L = the actual distance of the lights from each other;
A = the distance of the point midway between the lights from the surface;

the radius of curvature r of the surface, in the same unit as has been used for the above distances, is—

$$r = \frac{2AL}{L - 2l} \text{ for a convex, or}$$

$$r = \frac{2AL}{L + 2l} \text{ for a concave surface.}$$

The less the curvature, the greater must be the distance A, in order for the formulæ to hold good.

For lights, the flat flames of petroleum or gas lamps are very convenient if the edge be turned to the reflecting surface. With but little error we may employ the bars of a window, in front of which the observer is placed with the telescope.

When the curvature of lenses is determined after this manner, there are usually images reflected from the second side. In the case of biconvex or biconcave lenses, it is easily seen which are the images to be employed from their erect or inverted position.

The radius of curvature of a concave surface may also be determined by taking double its measured focal length (44).

Proof of the above formula for a convex surface.—The line L joining the two lights gives an image at the distance a, behind the spherical surface, by the rule $\frac{1}{a} = \frac{1}{A} + \frac{2}{r}$. ($\frac{1}{2}r$ is the focal length). The length λ of this image is given by $\frac{\lambda}{L} = \frac{a}{A}$. From these two formulæ we find $a = \frac{Ar}{2A + r}$, $\lambda = \frac{Lr}{2A + r}$. The image appears pro-

jected upon the scale touching the surface of the length, $l = \lambda \frac{A}{A+a}$; from which, by substituting the above values of λ and a, $l = \frac{1}{2} \frac{rL}{A+r}$, or $r = \frac{2Al}{L-2l}$. In just the same way is found the formula for concave surfaces.

44.—The Focal Length of a Lens.

The focus of a lens is the point at which rays parallel to the axis on incidence cross after emergence. The distance of the focus from the lens is the focal length. In concave lenses the focal length has the negative sign. The number of a spectacle lens is its focal length expressed in inches.

The two radii of curvature, r and r' of a lens, and the focal length, are related to each other and the refractive index of the sort of glass as follows :—

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} + \frac{1}{r'}\right), \text{ or } \mu = \frac{1}{f}\frac{rr'}{r+r'} + 1.$$

When a surface is concave its radius of curvature must be taken as negative.

(1.) The focal length of a convex lens may be measured by forming with it an image of the sun on a plate of ground glass, and holding it at such a distance that the image is sharp and clear. The distance of the plate from the lens is the focal length.

(2.) Or the lens is placed before the object-glass of a telescope which has previously been focussed on some very distant object. Looking, now, through the lens with the telescope at some plane object (*e.g.* a sheet of paper with writing on it), it will, a certain distance from the lens, be clearly visible. This distance is the required focal length.

(3.) The image formed by a lens of a near object may also be used to determine the focal length. On one side of the lens is placed a light, and on the other a white screen at such a distance that a clear image of the light is formed upon it. Taking a and b as the distances of the light and the image from the lens, and f the required focal length—

$$\frac{1}{f}=\frac{1}{a}+\frac{1}{b}, \text{ or } f=\frac{ab}{a+b}.$$

(4.) When the size of the image is the same as that of the object, both the image and the object are at a distance from the lens of double the focal length. (See 5.)

(5.) The methods given above assume that the thickness of the lens is so small in proportion to the focal length that it may be neglected. When this is not the case, we understand by the focal length the distance of the point of convergence of rays falling parallel on the lens from the *principal plane* of the lens or system of lenses. The principal plane may be found by construction as follows:—If the lines of incidence and emergence of a ray falling on the lens parallel to its axis are produced till they cut each other within the lens, the point of intersection lies in the principal plane which is perpendicular to the axis. But, without knowing the principal plane, the focal length of a thick lens or system of lenses may be found as follows:—On one side of the lens a brightly-illuminated scale is placed, a little farther from it than the focal length (the scale is best of glass with transmitted light). On the other side of the lens a white screen is placed at such a distance that a clear and greatly magnified image of the scale appears upon it. Then taking

l, the length of a division of the scale;
L, the length of its image;
A, the distance of the screen from the lens;

the required focal length f is—

$$f=A\frac{l}{L+l}.$$

Conversely, also, we may place a sharply-defined object at a great distance from the lens, and measure its image, now much diminished, on the other side of the lens. For this purpose it is best to use a scale divided on glass, read by means of a lens. It must be so placed that the divisions on the glass and the image of the object are clearly visible

through the lens. We must then take, in the previous formula, l for the length of the image, and L for that of the object, and A for the distance of the latter from the lens.

Proof.—The distances A and a of the image and the object from the principal plane of the lens are connected by the formula, $\frac{1}{A}+\frac{1}{a}=\frac{1}{f}$. Their magnitudes are in the ratio $\frac{L}{l}=\frac{A}{a}$. By putting $\frac{1}{a}=\frac{L}{Al}$ in the first equation, we get the expression as above. Since A is large compared with the thickness of the lens, we may, instead of the unknown distance from the principal plane, use the distance from the lens.

(6.) A concave lens which gives no real image, *i.e.* which has a negative focus, is used in combination with a stronger convex lens of known focal length, and the common focus of the two determined by one of the methods (1) to (4). If, then,

F = the common focal length ;
F' = that of the convex lens alone ;

the local length f of the concave lens will be found by

$$\frac{1}{f}=\frac{1}{F}-\frac{1}{F'}, \text{ or } f=\frac{FF'}{F'-F}.$$

(7.) Finally, the focal length of a concave lens may also be obtained by measuring the circle of light formed by the diverging rays of the sun when thrown on a screen at a given distance. For let

d = the diameter of the aperture of the lens ;
D = the diameter of the circle of light ;
A = the distance of the screen from the lens ;

and we have for the focal length—

$$f=\frac{Ad}{d-D+0{\cdot}0094A};$$

0·0094 is twice the tangent of the apparent diameter of the sun. When the lens is deep and not too small, this last

term may be neglected, and we thence obtain the simple rule, to take for the focal length that distance at which the circle of light on the screen is double the aperture of the lens.

45.—Magnifying Power of an Optical Instrument.

I. *Lens.*

The magnifying power of a lens is calculated from the focal length, which, for a thick lens or a combination of lenses, must be determined by (5) of the previous article. Calling

f = the focal length;
A = the least distance of distinct vision with the naked eye;

the magnifying power of the lens is—

$$m = \frac{A}{f} + 1.$$

For ordinary eyes the least distance of distinct vision may be taken as 25 cm.

Proof.—If a small object of the length l be placed at a distance a under the lens, so that its (virtual) image appears at the distance A, we have $\frac{1}{a} = \frac{1}{A} + \frac{1}{f}$. Let the image have the length L, and the magnification will be, $\frac{L}{l} = \frac{A}{a} = 1 + \frac{A}{f}$

II. *Telescope.*

The magnifying power is the ratio of the angle which a distant object subtends, seen through the telescope, compared with that which it subtends as seen with the naked eye.

(1.) The following method is universally applicable for determining the magnifying power. The telescope is placed at a distance, great compared with its length, before a measuring rod (a paper scale, a slated roof, or the pattern of a wall, will answer the purpose), on which two points must be sufficiently marked to be seen with the naked eye. Looking

now at the scale through the telescope with *one* eye, and with the *other* unassisted, the two images are seen superposed. If, then, the distance between the two points appear to correspond to n divisions of the scale, as seen through the telescope, while the actual distance is N divisions, the magnifying power is

$$m = \frac{N}{n}.$$

The observation will be much facilitated by drawing out the eye-piece of the telescope, so that the two images are not displaced relatively to each other by a movement of the axis of the eye. A short-sighted eye must, of course, be assisted by spectacles.

(2.) In telescopes with convex eye-pieces the following simple method is almost always applicable:—Firstly, the telescope must be so far drawn out that a distant object is clearly seen. The object-glass is then taken out and replaced by a screen with a narrow opening (a rectangle cut out of card-board), or by a transparent scale. The remaining lenses of the telescope will form a real image of the screen or scale. The ratio of the size of the object by which the object-glass has been replaced to that of the image is the required magnifying power.

To carry out this measurement we may employ a little transparent scale with a lens attached. This must be brought before the eye-piece, so that both the graduation on it and the image of the screen, or of the scale in the place of the object-glass, are clearly visible.

The circular opening of the cell of the object-glass itself may be used instead of the above-mentioned screen, if we are certain that the rays coming from its edges are not cut off by the diaphragms of the tube, as is frequently the case. A screen of angular form shows at once if this is the case.

Proof for Keppler's Telescope.—If F be the focal length of the objective, f that of the eye-piece, the magnifying power is, as is well known, $\frac{F}{f}$. The distance of the eye-piece from the objective

is, when a distant object is distinctly seen, $A = F + f$. The object of the length L in the place of the object-glass gives therefore an image of the length $l = \frac{fL}{A - f} = \frac{F}{fL}$ (see previous article, No. 5). Therefore $\frac{L}{l} = \frac{F}{f}$.

(3.) The focal lengths of the separate lenses and distances being known, the power may be calculated. For example, that of an astronomical telescope with a simple eye-piece, or of a Galilean telescope, is the focal length of the objective divided by that of the eye-piece. Practically, the results of these and similar rules are of little use, except to the optician making a telescope, since the focal length of the Galilean eye-piece cannot be measured directly, and telescopes with convex lenses are mostly of a compound nature. The exact measurement of the distances, frequently very small, between the lenses in the eye-piece offers great difficulties; and besides this, without determining the position of the principal plane, only a rough result can be obtained from the formulæ.

(4.) The size of the field of view is the angle made by the rays from two points of a distant object, the images of which are at the edges of the field of view diametrically opposite to each other. If l be the actual distance of these points from each other, and a their distance from the telescope, the size of the field of view is, expressed in degrees,

$$= 57^\circ{\cdot}3 \,.\, \frac{l}{a}\,.$$

In practice a measuring-rod fixed at some distance is again the most convenient object to employ.

III. *Microscope.*

(1.) The magnifying power of a microscope may be taken as the ratio of the angle under which a small object is seen in the microscope, to that which the same object would subtend at the smallest distance of distinct vision, which on the average we may take as 25 cm.

The method of determining the power of a microscope is

the same as that described in No. II. (1) for the telescope. An object, the length of which is known, is brought under the microscope, most conveniently again a small divided scale. At 25 cm. below the eye-piece is fastened a rule. Whilst one eye sees the object through the microscope, the other glances at the rule and measures upon it the projection of the image visible in the microscope. If the image appear N divisions in length, whilst its actual size is n divisions, the power is $\frac{N}{n}$.

A camera lucida, or small mirror, the silvering of which has been rubbed off in the middle, may also be fixed over the eye-piece, and the scale set up 25 cm. to one side, so that with the same eye the object under the microscope and the reflected image of the scale are both visible at once.

(2.) If a microscope be employed for measuring small lengths by means of a micrometer scale of known value, it becomes necessary to know, besides the above-mentioned power, the ratio of the length of the real image as formed on the micrometer to the length of the object. The measurement of this ratio is easily accomplished by the aid of a second micrometer scale of similar scale-value which serves as an object. The number of divisions of the micrometer scale which are covered by one division of the scale below is the required number.

For the purpose of microscope measurement of lengths, it is not necessary to know the actual size of the micrometer scale in the eye-piece. Its relation to that of the object may be much more directly determined by employing for the latter, once for all, an object of known length (a measuring scale).

In these measurements of length microscopically, it must not be overlooked that the power is altered by any change in the relative position of the eye-piece and objective. The eye-piece used for measurement must therefore always have the same position in the tube.

46.—Saccharimetry. Determination of the Rotating Power.

The angle α through which the plane of polarisation of the light is rotated by a solution of sugar, which contains z grms. of sugar in 1 c.c., is in a column l mm. long (according to Wild)—
for the yellow light of the sodium flame—

$$\alpha = 0^\circ{\cdot}6642 \,.\, z \,.\, l, \text{ whence } z = 1{\cdot}5056 \frac{\alpha}{l};$$

for white light (average)—

$$\alpha = 0^\circ{\cdot}7102 \,.\, z \,.\, l, \text{ whence } z = 1{\cdot}4080 \frac{\alpha}{l}.$$

The rotation is "right-handed," *i.e.* in the opposite direction to the spiral of a cork-screw. As to the use of the different instruments, the following remarks should be noticed:—

I. *Mitscherlich's Saccharimeter.*

A sodium flame (Berzelius' lamp with common salt on the wick, or a Bunsen's burner with a soda bead on platinum wire, taking care to exclude the light from the ignited bead) is placed behind the instrument, in front of a black screen. Then a tube, empty or filled with pure water, is placed between the Nicol's prisms of the instrument, and the index turned over the circle nearest the eye until the middle of the field of view appears dark. The tube is then filled with the solution of sugar and put into its place again. The field of view with the first position of the index, appears bright. The number of degrees through which the index must be turned to the right (in the direction of the hands of a watch), that the centre of the field may be dark again, is the required angle of rotation α.

If we intend the zero of the circle to be also the point from which the angle is measured, the index is put to the

zero without any sugar solution, and the farther Nicol turned until the centre of the field is dark.

There are always two positions where the darkening takes place, 180° distant from each other.

II. *Wild's Polaristroboometer.*

In this instrument bands are seen in the field of view, which are bright and dark with homogeneous (sodium) light, but coloured when white light is used. The eye-piece is first so far pulled out that the bands are as sharp as possible.

The adjustment for saccharimetry is, just as in I. for the darkening of the field, so here for the disappearance of the bands in the middle of the field of view. Since it is the Nicol's prism nearest to the eye which is turned, the rotation, as seen by the eye, must be taken in the direction contrary to that of the hands of a watch.

The bands disappear in four positions, 90° from each other. The measurement is made more accurate by observing in all the four quadrants, both with and without the sugar-solution.

The question which sometimes arises whether the angle of rotation α is greater or less than 90°, may be answered by making an approximate determination of the specific gravity and the use of Table 3, or else by making a second observation with a tube of a different length.

The instruments frequently have a second graduation, which, when using a tube 200 mm. long, gives at once the sugar contained in 1 litre of the solution in grammes.

It is sufficiently evident how the rotating power of any other substance is determined with either Mitscherlich's or Wild's apparatus.

III. *Soleil's Saccharimeter.*

This has behind the tube a double plate consisting of two semicircles of quartz, one rotating the plane to the left and one to the right. The eye-tube is first so far pulled out,

that, using the ordinary lamp, the two semicircles are sharply defined. The adjustment is made to the same colours in the two semicircles, and usually the "sensitive" transition tint from blue to red is chosen. In order to obtain this, nearly the same colours are produced, either by using the rack on the eye-piece or by rotating the hinder (analysing) Nicol's prism. By turning the tube in the eye-piece any desired colour may then be obtained, and that is chosen which gives the greatest difference of tint between the semicircles.

In Soleil's instruments the divided circle is replaced by a wedge of quartz, which is moved by a spring; the amount of motion is read off on a little scale with an index. It will be sufficient to remark here that the motion through 1 division corresponds to a revolution of the yellow sodium light—in the Paris instruments (Soleil Duboscq)

$$\text{of } 0^{\circ}\cdot217\,;$$

in the Berlin instruments (Soleil Ventzke)

$$\text{of } 0^{\circ}\cdot346.$$

The sugar contained in 1 c.c. of the solution in grammes, will therefore be, using a tube 200 mm. long, when the displacement from the position when the tube is empty is p divisions—

$$\begin{array}{ll} \text{Soleil Duboscq} & z = 0\cdot1635\,.\,p, \\ \text{Soleil Ventzke} & z = 0\cdot2605\,.\,p. \end{array}$$

For specimens of sugar, therefore, in which the percentage of pure sugar is to be determined, the rule is: dissolve 16·35 (or 26·05) grms. of the sugar to 100 c.c. of the solution; the displacement then gives the percentage of pure sugar.

To test the accuracy of the divisions, a "normal solution" of pure sugar containing 16·35 (or 26·05) grms. in 100 c.c. is used. The displacement must amount to 100 divisions.

If we wish the zero of the divisions to coincide with no sugar in the solution, the index is placed at the zero, when the empty tube is in its place, and the analysing prism rotated until the semicircles have the same colour.

Determination of Sugar in the presence of other optically-active Substances.—The elimination of the influence of other optically-active substances, besides cane-sugar (*e.g.* inverted sugar or dextrin), depends upon the fact that the cane-sugar, rotating the plane of polarisation to the right, is, by warming for 10 minutes to about 70° with hydrochloric acid, changed into inverted sugar, which has left-handed rotatory power. An inverted solution l mm. long, which contains in 1 c.c. z grms. of what was cane-sugar, rotates the plane of polarisation of the sodium flame, at the temperature t', through the angle

$$\alpha' = (0°{\cdot}2933 - 0°{\cdot}00336t')\, z \, . \, l \, .$$

Hence the practical rule:—After the angle α has been determined with the usual solution, 100 c.c. of the solution are taken, mixed with 10 c.c. of strong hydrochloric acid, and kept for 10 minutes at a temperature of 70°. When the fluid has cooled, a tube one-tenth longer than the former one is taken (or if the same tube be used, the angle obtained must be multiplied by 1·1), and the rotation to the left, α', which now is produced, is observed. Let the temperature of the solution at this latter observation be t'. The angle is then calculated

$$\frac{\alpha + \alpha'}{1{\cdot}442 - 0{\cdot}00506t'}$$

and from this the sugar originally contained is got by the instructions given for the particular instrument used.

47.—Angular Measurement with Telescope, Mirror, and Scale.

This method may be employed with great advantage in many magnetic and galvanic observations, but its application is limited to the measurement of small angles.

A small vertical mirror is attached to the suspended magnet, etc., of which the horizontal deflection is to be measured, and, in order to simplify calculation, should be near the axis of rotation of the latter. At a distance of from 1 to 5 metres from the magnet is fixed a horizontal scale at the same level as the mirror; in which its reflected image is observed with

a telescope provided with cross-wires. The scale must be so placed that when the magnet is in its position of equilibrium, to which the other positions are mostly referred, that point of the scale from which a perpendicular would cut the mirror shall be visible on the cross-wires of the telescope. We may call this point briefly the "middle scale-division," and the corresponding position of the magnet its "mean position."

Arrangement of the Telescope and Scale.—The telescope is first focussed approximately for twice the distance between the mirror and scale. It is then pointed to the mirror, and so placed that its objective is visible, in the mirror, to an eye immediately over the middle scale-division, or conversely that this is seen from near the telescope. The image of the scale will then be visible in the telescope, or will appear by a slight movement of the latter. Lastly, the fine adjustments must be made; the cross-wires must be clearly focussed, and the telescope then drawn out till the scale and cross show no parallax; that is, till their relative position is unaltered by moving the eye before the eye-piece.

If observers requiring different foci take turns at reading, clear definition must be obtained in each case by moving only the eye-piece between the eye and the cross-wires.

Note.—In many cases a lamp with a narrow flame may be advantageously substituted for the telescope; a lens of focal length equal to the distance between the scale and mirror being placed immediately in front of the latter, or a concave mirror being employed. An image of the flame will then be projected on the scale, and serve as an index. For galvanometric observations this is especially useful, as it can be seen from anywhere in front of the scale, and by many people at once. The calculations of arc, etc., are precisely the same as with the telescope.—*Trans.*

Receipt for Silvering Glass (after Boettger).

(1.) Argentic nitrate is dissolved in distilled water, and ammo nia added to the solution till the precipitate first thrown down is almost entirely redissolved. The solution is filtered and diluted, so that 100 c.c. contain 1 grm. of argentic nitrate.

(2.) 2 grms. of argentic nitrate are dissolved in a little water, and poured into a litre of boiling water; 1·66 grm. of Rochelle

salt is added, and the mixture is boiled for a short time, till the precipitate contained in it becomes grey, and is then filtered hot.

The glass plates, thoroughly cleaned (with nitric acid, caustic soda, alcohol), are placed in a shallow vessel, and covered a few millimetres deep with equal volumes of the two solutions. In an hour the reduction will be complete; the plates are washed and the operation repeated until a sufficient coating of silver is obtained. When the silvered surfaces are dry, they may be cautiously polished with the palm of the hand. If the silver be only required as a coating of the back surface, the polishing is of course superfluous. In this case also the operation may be shortened by heating the solutions to about 70° C. before mixing. The silver may then be varnished over as a protection.

The properly-prepared solutions will keep for about a month in a dark place. The thin glasses used for covering microscopic objects make good mirrors, but those only which give a clear image of the scale can be employed.

48.—Reduction of Observations with the Scale to Angular Measure.

We will reckon all angles of rotation from the "mean position" (see above), as zero, and denote by φ the angle of deflection through which the magnet, etc., is turned from this position. As scale-deflection, we take the difference n of the observed from the middle scale-division.

(1.) For small deflections the angle is proportional to the scale-reading; and, indeed, if r be the distance of the reflecting surface from the scale, expressed in scale-divisions, (millimetres, if it be a millimetre scale), the value of 1 division in degrees of arc is—

$$= \frac{28^\circ{\cdot}648}{r} = \frac{1718'{\cdot}9}{r} = \frac{103132''}{r}$$

In observations of the variation of terrestrial magnetism, for instance, this proportionality may always be assumed.

The error may amount at most

in deflections of	1°	2°	3°	4°	5°
in parts of the whole to	0·0004	0·0016	0·0036	0·0064	0·010

(2.) For a deflection not exceeding 6° we may always with sufficient accuracy take—

$$\varphi = \frac{1718'.9}{r} n \left(1 - \tfrac{1}{3}\frac{n^2}{r^2}\right)$$

Frequently a trigonometrical function is required instead of the angle itself—

$$\tan\varphi = \frac{n}{2r}\left[1 - \left(\frac{n}{2r}\right)^2\right]$$

$$\sin\varphi = \frac{n}{2r}\left[1 - \tfrac{3}{2}\left(\frac{n}{2r}\right)^2\right]$$

$$\sin\frac{\varphi}{2} = \frac{n}{4r}\left[1 - \frac{11}{2}\left(\frac{n}{4r}\right)^2\right]$$

Hence we reduce a scale-reading n to the corresponding arc, tangent, sine, and sine of half-angle, by subtracting $\frac{1}{3}$, $\frac{1}{4}$, $\frac{3}{8}$, or $\frac{11}{32}\,\frac{n^3}{r^2}$ respectively from n.

(3.) For considerable deflections—

$$\varphi = \tfrac{1}{2}\tan^{-1}\frac{n}{r}$$

The last formula is given by simple geometrical considerations, the others by taking the first two terms only of the series for the development of φ, $\tan\varphi$, etc.

49.—Determination of the Position of Equilibrium of a Swinging Magnetic Needle.

The *position of equilibrium*, or point of the scale on which a magnetic needle would settle when it came to rest, may be determined from observations of the moving needle in the following manner:—

1. *Observation of Turning-Point.*—If the oscillations are rapid or large, a number of successive turning-points of the cross-wires (*i.e.* points at which the direction of motion is reversed) are observed on the scale. From any three of these the position of equilibrium may be found by taking the arithmetical mean of the first and third, and again that of the second and the number so obtained. (Compare, besides, the article (7) on the determination of the position of equilibrium of a balance, which is completely applicable to the present case.

(2.) *Observation of Position.*—If the motion of the needle be so slow that the position of the cross-wires on the scale can be exactly observed at any moment, the arithmetical mean between two successive readings, differing by the time of one oscillation, will give the position of equilibrium. As an example we may take a set of observations of the position of a magnet, observed and calculated after the directions given by Gauss. The position of a needle, of which the time of oscillation is 20 seconds, is required for 10 hours 0 minutes, p is the reading at each successive 10 seconds, p_0 the means between each pair of p differing by 20 seconds.

Time.			p	p_0	Mean of p_0.
hrs.	mins.	secs.			
9	59	30	475·0		
		40	474·8	475·50	
		50	476·0	5·95	
10	0	0	477·1	6·40	476·28
		10	476·8	6·60	
		20	476·1	6·95	
		30	477·1		

(3.) *Damped Magnetic Needles.*—These two rules are only applicable when the amplitude of swing diminishes very slowly. If, however, the magnet be damped and rapidly brought to rest (by the employment of a copper frame), the position of equilibrium p_0 is found from two successive observations, p_1 and p_2, differing by the time of one oscillation, by the following formula:—

$$p_0 = p_2 + \frac{p_1 - p_2}{1 + k}.$$

Here k is the "ratio of damping," that is, the ratio of one arc of oscillation to the next following. Compare the example in the following article. The reduction of scale-readings to angular measure is rarely necessary.

To bring the needle to rest, a magnet is frequently employed, which is approached or withdrawn at the same level as the needle. A galvanic current passing near the needle, and closed and broken at the right moments, may be employed for the same purpose.

50.—Damping and Logarithmic Decrement of a Magnetic Needle.

The diminution of the arcs of oscillation of a magnetic needle which is damped by a copper case, or by the surrounding coils of a multiplier, is of great importance in galvanic and magnetic measurements. The damping is caused by the reaction of the currents induced in the neighbouring conductors by the moving needle; and the law of damping, given by the theory of induction, shows that the arcs diminish in a geometrical series. The constant relation of an arc of oscillation to that next following is called the ratio of damping, and the logarithm of the latter, the logarithmic decrement of the needle.

The determination of this magnitude is most simply accomplished by observation of a series of turning-points of the needle. The difference of two successive turning-points, which, if the oscillations be large, must be corrected to angular measure (article 48), gives the arc. If a_m be the amplitude of the mth, and a_n that of the nth oscillation, the ratio of damping—

$$k = \left(\frac{a_m}{a_n}\right) \frac{1}{n-m};$$

and the logarithmic decrement—

$$\lambda = \frac{\log a_m - \log a_n}{n-m}.$$

Errors of observation have the feeblest influence on the result when $\frac{a_m}{a_n}$ is about 3.

From a long series of observations (best an uneven number) the required magnitude may be deduced as in the following example, in which 7 observations of turning-point are contained in the first column. The second column gives the distance of the turning-point from the middle scale-division (in this case 500); the third and fourth the correction (art. 48), to reduce the scale-readings to numbers proportional

to their angular values. The distance of the scale from the mirror is $r = 2600$ scale-divisions. In column 5 are the six corrected arcs, of which combinations of the first and fourth, second and fifth, etc., each give a value for k and λ. In the sixth and seventh columns is shown the method (49, 3) of calculating the position of equilibrium from the two turning-points when the ratio of damping, $k = 1{\cdot}151$, is known.

Example—

Observed turning-points.	n	$\frac{n^3}{3 \times 2600^2}$	Corrected turning-points.	Arcs a	$\frac{a}{2{\cdot}151}$	Position of Equilibrium.
285·0	215	0·5	285·5			
710·0	210	0·5	709·5	424·0	197·1	512·4
341·2	159	0·2	341·4	368·1	171·1	512·5
662·5	162	0·2	662·3	320·9	149·2	513·1
383·9	116	0·1	384·0	278·3	129·4	513·4
625·7	126	0·1	625·6	241·6	112·3	513·3
415·6	84	0·0	415·6	210·0	97·6	513·2
						mean = 513·09

We obtain also—

from 1 and 4, $\lambda = \frac{1}{3}$ (*log* 424·0 − *log* 278·3) = 0·0610
" 2 " 5, " 368·1 241·6 0·0609
" 3 " 6, " 320·9 210·0 0·0614
mean $\lambda \doteq$ 0·0611
k = 1·151

A part of the damping is always dependent on the resistance of the air. If the damping be required which is due to the multiplier alone, one series of observations must be made with open, and another with closed circuit. The logarithmic decrement of the former subtracted from that of the latter gives the required decrement due to the multiplier alone. (70, III.)

51.—Time of Oscillation of a Magnetic Needle.

The time of oscillation of a body oscillating about its position of equilibrium is the time between one elongation (turning back, greatest deviation from the point of rest) and the next on the opposite side. The instant of turning, however, is unsuitable for direct observation, as at that moment

the motion of the body is insensible. It passes a point near its position of equilibrium with the greater velocity, so that the instant of this crossing may be exactly observed. From the times of two successive passings of the same point in opposite directions, that of the intermediate turning-point is simply found as the arithmetical mean.

A point near the position of equilibrium is marked on the scale (by hanging over it a dark thread), and the times at which it is passed are observed by the ticking of a seconds clock. The mean of each successive pair of observations is taken, and the differences between these means give the time of oscillation. The tenths of seconds are estimated by the relative distances of the cross-wires from the mark at the ticks of the clock preceding and following the passing.

If from a consecutive series of n values so obtained the mean be again taken, it will only yield the same result as if the difference of time between the first and last observation were divided by n. The intermediate observations will therefore be useless. To render the whole available, the observations may be divided into two parts, and the differences of the corresponding numbers in the two halves taken, and from these the arithmetical mean reckoned and divided by $\frac{1}{2}n$.

Example—

Time of crossing (observed).		Time of turning (reckoned).		Time of oscillation.
m.	sec.	m.	sec.	sec.
10	3·3			
	16·5	10	9·90	from Nos. 1 and 4, $\frac{39{\cdot}90}{3} = 13{\cdot}30$
	29·9		23·20	
	43·0		36·45	„ 2 „ 5, $\frac{40{\cdot}05}{3} = 13{\cdot}35$
	56·6		49·80	
11	9·9	11	3·25	„ 3 „ 6, $\frac{40{\cdot}15}{3} = 13{\cdot}38$
	23·3		16·60	
				mean, 13·34

It is best of all to obtain two widely-separated and exactly-determined times of elongation from repeated observations, in the following manner :—We observe twice (or for great accuracy even more frequently) an even number of

successive times of passing the marked point, and from each pair lying symmetrically about the middle elongation we take the arithmetical mean, and from these, again, the mean of the whole.

Example—

	First Set.			Second Set.	
No.	Times of passing.		Means.	Times of passing.	Means.
	m. sec.		m. sec.	m. sec.	m. sec.
1.	4 10·6			9 25·5	
2.	29·0			43·9	
3.	45·6	3 . 4	4 54·80	10 0·6	10 9·75
4.	5 4·0	2 . 5	54·85	18·9	9·75
5.	20·7	1 . 6	54.75	35·6	9·70
6.	38·9			53·9	
	Mean of whole		4 54·80		10 9·73

These two means are the times of two elongations, as exactly as they can be deduced from these observations. The difference between them, 314·93 sec., divided by the number of intermediate oscillations, gives the time of oscillation with the greatest accuracy. It is not necessary actually to count these oscillations, as the number may be deduced from the observations themselves. An approximation to the time of oscillation is easily obtained from either series. Taking, for example, the first: from the first and last pairs of observations are obtained the times of two elongations—viz. 4 m. 19·8 sec. and 5 m. 29·8 sec., between which four oscillations have occurred. Hence the time of oscillation is $\frac{70\cdot0}{4} = 17\cdot5$ sec. If this number and the observations were perfectly exact, 17·5 would divide 314·93 without a remainder, and the quotient would be the number of oscillations sought. Performing the division we find 17·995, a value so near to the whole number 18 as to leave no doubt that this is the number of oscillations in 314·93 sec. The exact time of oscillation is therefore $\frac{314\cdot93}{18} = 17\cdot496$. Were 17 or 19 taken, 18·52 or 16·57 respectively would be obtained as the period, and neither of these agrees with the result of the single observations.

In order to eliminate errors of observation, a large even number, $2m$, of sets of observations may be made, and No. 1 combined with $m + 1$, 2 with $m + 2$ . . . . m with $2m$, and the mean of the single results taken. If the sets of observations are separated by equal intervals, the method of least squares may be employed; exactly as in article 37.

This method obviously requires that the oscillations should be sufficiently slow for the time of each to be observed. It may, however, be employed for more rapid oscillations, by each time omitting 2 (or any even number of) passings, and forming the set, for instance, of Nos. 1, 4, 7, 10, 13, and 16, which are reckoned precisely as above, except that the result is of course divided by 3. In the estimation of the number of oscillations between the sets, the care required will naturally increase with the number, and, other things being equal, with the rapidity of the oscillation. The possibility of an error will be diminished if we observe at each passage whether the motion corresponds to a greater or lesser period, and also by our accustoming ourselves always to begin with a passage in the same direction. The required number of oscillations will then necessarily be even.

It is manifestly unimportant to the method whether the observations are made with mirror and scale, or with the naked eye.

If the time of oscillation be very near a second, or an exact multiple or sub-multiple of one, the method of coincidences may be employed. In this case, the times must be noted at which the passage of the position of equilibrium exactly coincides with the beat of a seconds clock. The time of oscillation is then given by dividing the number n of seconds between two such coincidences by $n + 1$ or $n - 1$, according to whether the oscillations are quicker or slower than those of the pendulum.

If a watch or chronometer be employed instead of a clock, it is convenient to count 5 or 10 ticks after the passage of the marked division before noting the time, so as to allow time to look from the telescope to the watch. If an absolute time be wanted, this must, of course, be subtracted from the mean result. A spot of light reflected on the scale from a lamp (as in Thomson's galvanometers) is often conveniently substituted for the telescope.

52.—Reduction of the Time of Oscillation to that in an infinitely Small Arc.

The time of oscillation increases slightly with the amplitude. As we usually require the limiting value to which the time approaches when the oscillations are very small, we must apply a correction to the observed values which are obtained from larger amplitudes.

Taking

t = the observed period of oscillation ;
α = the arc through which the magnet vibrates ;

the time of oscillation, in an infinitely small arc, is—

$$t_0 = t - \left(\tfrac{1}{4} \sin^2 \frac{\alpha}{4} + \frac{5}{64} \sin^4 \frac{\alpha}{4}\right) t.$$

To facilitate the calculation, the quantity within the brackets may be found in Table 21, calculated for arcs up to 40° ; an amplitude which should never be exceeded.

This correction is applicable to a pendulum moved by its weight, or to any oscillating body in which the force which draws it towards its position of equilibrium is proportional to the sine of its angle of displacement.

The method of observation with the telescope and scale possesses the advantage that the oscillations (of from 50 to 800 divisions of the scale) are so small that the first term of the formula of correction is sufficient. We may therefore write, if

p = the arc of oscillation in divisions of scale ;
r = the distance of mirror from scale ; also expressed in divisions of scale—

$$t_0 = t - \frac{t}{256} \frac{p^2}{r^2}.$$

The value of α (or p, as it is written in the above formula) may be taken as the mean of the arcs of the first and last observed oscillations. The observations must be so arranged that the amplitude does not diminish by more than one-third during the experiment.

If we call the mean of the first and last arcs of oscillation a, and their difference d, α or p is more exactly

$$a\left(1-\frac{1}{24}\,\frac{d^2}{a^2}\right).$$

The complete formula for reduction of time of oscillation to that in an infinitely small arc is—

$$t_{\circ}=\frac{t}{1+\frac{1}{4}\sin^2\frac{\alpha}{4}+\frac{9}{64}\sin^4\frac{\alpha}{4}\,.\,.\,.\,.\,.}$$

The formula given above is obtained from this by performing the division, omitting all powers beyond the 4th, which is practically always admissible. The reduction formula for scale observations may readily be found with the help of article 48.

53.—Determination of Moment of Inertia of a Body.

The moment of inertia of a material point, referred to an axis round which it revolves, is l^2m, where $m=$ the mass of the point, and l its distance from the axis. That of a number of points rigidly connected, or of a body, is the sum or integral of those of all the individual points. It must of course be expressed by some units of length and mass. In the following cases lengths are always given in millimetres, and masses in milligrammes. We may indicate this by affixing to the number the sign Mgr. Mm².

I. *Calculation of Moment of Inertia.*—In bodies of regular form and even thickness the moment of inertia may be found by calculation.

In the following formulæ, which embrace the more frequent cases, m is always the mass of the body, and K its required moment of inertia.

Thin bar of length l, and of thickness uniform, and very small compared to l. Referred to an axis at right angles to the rod, and passing through its centre—

$$K=m\,.\,\frac{l^2}{12}\,.$$

Right-angled parallelopipedon.—a and b are two adjacent edges. The moment of inertia round an axis passing through the centre of gravity, and parallel to the third edge (that is perpendicular to a and b), is—

$$K = m\,\frac{a^2 + b^2}{12}$$

Cylinder of radius r referred to the axis of the cylinder—

$$K = m\,\frac{r^2}{2}$$

Referred to an axis perpendicular to the middle of the axis of the cylinder (l being the length of the cylinder)—

$$K = m\left(\frac{l^2}{12} + \frac{r^2}{4}\right)$$

Hollow cylinder of r_0 inner, and r outer radius. Moment of inertia referred to the axis—

$$K = m\,\frac{r_0^2 + r_1^2}{2}$$

Sphere of radius r, referred to a diameter—

$$K = m\,\frac{2}{5}\,r^2$$

Example.—The moment of inertia of a magnet 100 mm. long, 6 mm. radius, and weighing 88030 mgr.,

$$\text{is } 88030\left(\frac{100^2}{12} + \frac{6^2}{4}\right) = 74150000 \text{ mgr. mm}^2.$$

Note.—If, as in the foregoing examples, the moment of inertia K, relative to an axis passing through the centre of gravity, be given, the moment of inertia K, relative to another axis parallel to the first, may be obtained by adding to K the product of the mass of the body m and the square of the distance a between the new axis and the centre of gravity; that is—

$$K_1 = K + a^2m.$$

II. *Experimental Determination of the Moment of Inertia.*—The moment of inertia may always be found experimentally

in the following manner. The time of oscillation must be observed, and the moment of inertia then increased by a known amount, without altering the directive force, and the time of oscillation observed again. If

t = the time of oscillation of the body alone;
t' = the time with the added weight;
k = the moment of inertia;

both times of oscillation being reduced to an infinitely small arc; then—

$$t'^2 : t^2 = (K + k) : K$$

and the required moment (see preceding article) of inertia of the body alone—

$$K = k \frac{t^2}{t'^2 - t^2}$$

This method is specially applicable to the bodies hung by a thread, so as to turn about a vertical axis, and particularly to magnets. The known moment of inertia may be added by hanging two similar cylindrical weights upon points, or by threads, at equal distances from the axis of revolution (the suspending thread), and so that the axes of the cylinders are vertical. The turning force is unaltered by the added weight, as only the horizontal force is taken into consideration. The moment of inertia of the two cylindrical weights together is—

$$K = m\ (t^2 + \tfrac{1}{2}\ T^2)$$

m being the mass of both together, t the horizontal distance of the centres of suspension (points or threads) of the weights from that of the magnet (its axis of revolution), and r the radius of the cylinders.

l is determined by measuring the whole distance between the points of suspension of the weights, and halving it.

Example.—The two cylindrical weights are each 10 mm. in diameter $r = 5$
they weigh together 50 grm. $m = 50000$

The distance between the cocoon threads by which they are hung = 100·26 mm. $l = 50·13$.

Their united moment of inertia, therefore—

$$k = 50000 \left(50·13^2 + \frac{25}{2}\right) = 126280000 \text{ Mgr. Mm}^2.$$

Further, the time of oscillation is found to be—

1st, Of the unloaded magnet, 9·754 sec. in a mean arc of 18°·9; therefore (preceding article)—

$$t = 9754\ (1 - 0·00170) = 9·737$$

2d, Of the magnet loaded with the above weights, 14·311 sec. in an arc of 25°·5; therefore—

$$t' = 19\ .\ 311\ (1 - 0\ .\ 00310) = 14·267$$

Hence the required moment of inertia of the magnet—

$$K = k\frac{t^2}{t'^2 - t^2} = 126280000\ \frac{9·737^2}{14·267^2 - 9·737^2}$$
$$= 110110000 \text{ Mgr. Mm}^2.$$

54.—Coefficient of Torsion.

In order to make absolute measurements, it is necessary to separate the turning-force due to the elasticity of the suspending fibre of the magnet from that of the earth's magnetism. The moment of torsion of the thread is proportional to the imparted angle of torsion, while the directive force of the earth's magnetism is proportional to the sine of the angle which the magnet makes with the magnetic meridian, or very approximately to the angle itself, so long as it is very small. On this supposition, for an angle, the turning-force of torsion bears a certain ratio to that of the earth's magnetism, which we will call the ratio of torsion, and which is measured in the following manner:—

The position of the magnet, when the thread is not twisted, is first observed; then by turning the upper or lower points of attachment of the thread, a measured torsion is communicated to it, and the position of the magnet is again observed.

If α = the angle through which the thread is twisted;

φ = the angle through which the torsion deflects the magnet;

the required torsion ratio Θ is—

$$\Theta = \frac{\varphi}{\alpha - \varphi}.$$

In instruments for fine measurements the suspending fibre is attached, either above or below, to a graduated circle, by turning which any degree of torsion may be produced. The angle of rotation read on this circle is α. In the absence of such a circle the magnet must be turned once entirely round without moving the upper attachment of the thread; α will then be 360°.

For the sake of exactness it is desirable to make the observation with mirror and scale (art. 47), which is easily done by attaching a small mirror to the magnet, in case it is not already provided with it. If the angle of deflection be measured in degrees, of course the deflection of the magnet must also be reduced to the same unit.

55.—Magnetic Inclination.

Inclination is the angle which the direction of terrestrial magnetic force makes with the horizontal. This direction will be given by a magnetic needle if it is movable, without friction, on an axis at right angles to itself; and to the magnetic meridian, if (1) the axis passes through the centre of gravity of the needle; and (2) if its magnetic axis (the line uniting the two poles) is coincident with its geometrical axis. The impossibility of permanently satisfying these two conditions complicates the mode of observation described above.

The placing of the divided circle in the magnetic meridian is accomplished by the aid of an ordinary compass-needle, for which an accuracy within 1° or 2° is sufficient.

The numbering of the divisions of the circle varies in different instruments. It is most convenient when in each quadrant the divisions are numbered from the horizontal as zero; and for simplicity we will suppose, in the following, that this is the case. If the circle be movable, it must be numbered in this manner.

The vertical position of the circle is determined, in an instrument with fixed circle, by a plummet suspended from the uppermost 90° division; but in the best instruments with movable circles it is known to be the case when the bubble of a spirit-level retains the same place in its tube in all positions of the circle.

In each position of the needle both upper and lower points must be read, to eliminate any possible eccentricity of its axis from the centre of the circle. The mean of the two readings will in what follows be called shortly "the observed angle."

On account of possible lateral eccentricity of the centre of gravity, the needle must now be turned round (with a movable circle, the circle and needle together must be turned 180°), by which we also eliminate the deviation of the geometric from the magnetic axis of the needle (and, with movable circles, any deviation of the line joining the upper and lower 90° divisions from the axis of rotation of the instrument). Any longitudinal displacement of the position of the centre of gravity requires for its elimination a reversal of the magnetism of the needle.

We must observe the angles—

(1.) φ_1 in the first position of the needle.

(2.) ψ_1 when the needle is turned 180° round its magnetic axis, and again replaced in the instrument; or, with movable circle, when the latter, with the needle, is turned 180°.

(3.) φ_2 when the magnetism of the needle is reversed by stroking with a bar magnet in position 1.

(4.) ψ_2 when the remagnetised needle is placed in position 2, or the circle turned 180°.

I. If these angles are nearly alike, the inclination i is the arithmetical mean—

$$i = \frac{\varphi_1 + \psi_1 + \varphi_2 + \psi_2}{4}$$

II. In any case it may easily be managed by grinding the side of the needle before the observation that φ_1 and ψ_1, and also φ_2 and ψ_2 are nearly alike · and then—

$$tan\ i = \tfrac{1}{2}\left(tan\ \frac{\varphi_1 + \psi_1}{2} + tan\ \frac{\varphi_2 + \psi_2}{2}\right).$$

III. Should, however, φ_1 and ψ_1 also differ considerably, we must write—

$$cot\ \alpha_1 = \tfrac{1}{2}\ (cot\ \varphi_1 + cot\ \psi_1),$$
$$cot\ \alpha_2 = \tfrac{1}{2}\ (cot\ \varphi_2 + cot\ \psi_2);$$

and calculate lastly—

$$tan\ i = \tfrac{1}{2}\ (tan\ \alpha_1 + tan\ \alpha_2).$$

It is obvious that by reversing the needle any deviation of the magnetic from the geometrical axis of the needle will be eliminated. Formula I. also needs no remark. Formulæ II. and III. are obtained by supposing the unknown displacement of the centre of gravity resolved into its components, parallel and perpendicular to the magnetic axis of the needle, and considering the conditions of equilibrium between magnetic force and that of gravitation.

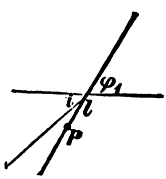


Fig. 9.

Were there, for instance, only a longitudinal displacement l of the centre of gravity towards the north pole of the needle, then taking φ_1 the observed angle, p the weight of the needle, M its magnetic movement, and T the total intensity of terrestrial magnetism, we should have—

$$pl\ cos\ \varphi_1 = MT\ sin\ (\varphi_1 - i)$$

If now the magnetisation of the needle be reversed, so that the displacement of the centre of gravity is towards the southern end, we have—

$$pl\ cos\ \varphi_2 = MT\ sin\ (i - \varphi_2)$$

By cross multiplication of these two equations, and division by $cos\ i\ cos\ \varphi_1\ cos\ \varphi_2$, we obtain—

$$tan\ i - tan\ \varphi_2 = tan\ \varphi_1 - tan\ i;$$

or (II.) $tan\ i = \tfrac{1}{2}\ (tan\ \varphi_1 + tan\ \varphi_2).$

It is assumed that the magnetic moment of the needle is the same before and after remagnetisation, which is very nearly the case if it be performed by carefully and equally stroking a thin and frequently remagnetised needle. It is well, to begin with, once to reverse the magnetism of a needle which has been long magnetised in one way.

Further, for observations φ_1 and φ_2, the needle is stroked exactly in the same way, but with the opposite poles of the stroking magnet.

The stroking itself is performed in the following manner:—Holding the needle by one half, with the fingers near the axis of rotation, the other half is drawn lengthwise over the pole of a magnet as in the figure. So, for instance, the two surfaces of one end should be twice gone over; then those of the other end four times, and then the first twice again.

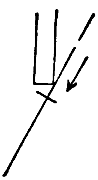


Fig. 10.

For measurement of inclination with the earth-inductor, see (77).

56.—Declination of Terrestrial Magnetism.

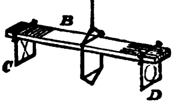
By "declination" is understood the angle which the magnet makes with the astronomical meridian; and to indicate the direction of the deflection, the angle is counted in the north, from the latter to the former. With us the declination is "west." The uncertainty of the position of the magnetic axis of a needle involves, for exact determination, an observation in two positions.

For the measurement (after Gauss) we require a theodolite with a horizontal circle, a distant mark (or if near, in the focus of a lens placed before it), of which the astronomical azimuth is known (that is, the horizontal angle which a straight line, drawn through it from the theodolite, forms with the astronomical meridian); and lastly, a magnetometer of which the needle can be turned upside down. The theodolite is placed nearly in the same magnetic meridian as the suspending thread of the needle, and its telescope at the same height as the magnet.

We assume, as is most convenient, that the magnet has a longitudinal sight, which at the end towards the theodolite has a lens of the same focal length as the magnet. At the other end is a mark (screen with small opening, cross-threads, or divided glass), which, seen through the lens, appears as a very distant object.

The theodolite is divided so that the number increases in turning the telescope the same way as the sun (that is, from left to right).

The observations, after the axis of the instrument is made vertical by the aid of the level, are as follows:—



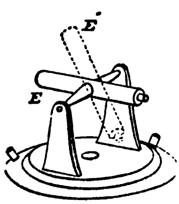


Fig. 10a.

(1.) The telescope is pointed so that the terrestrial mark appears on the cross-wires. The reading of the circle is now $= \alpha$. If A be the astronomical azimuth of the mark, counting from north to west (see above), the theodolite must be turned to the division $\alpha + A$ in order that the line of vision of the telescope may point north.

(2.) The telescope being directed to the mark on the magnet, the reading of the circle is α_1.

(3.) The magnet is turned on itself 180°, or so that the side is uppermost which was previously below, and the telescope directed again to its mark. The reading of the circle is α_2. The readings α_2 and α_1 always differ but very slightly.

Now clearly the declination will be—

$$\delta' = \alpha + A - \frac{\alpha_1 + \alpha_2}{2}$$

west, when the suspending thread has no torsion. To determine and eliminate the latter, we must measure the angle to which the thread has been twisted in the observation. For

this purpose the magnet must be taken from its stirrup, an unmagnetised bar of approximately equal weight substituted for it; and the turning of the stirrup by this change measured on a divided circle laid underneath. Should this angle of rotation $= \varphi$ in the same direction as the sun's daily course, the declination will be—

$$s = \delta' + \Theta\varphi \, ;$$

Θ being the ratio of torsion (54).

The smallest ratio of torsion in proportion to its strength is given by the cocoon thread, but its position of equilibrium of torsion is very changeable, and, in a bundle of fibres, dependent on the weight hung to it. Moreover, for small moments of torsion, the observation of angles of torsion is tedious and inexact, so that a metallic wire (thin iron or brass) answers best if the magnets are not too small.

57.—Surveying with the Compass.

The 23d Table contains the angles of deviation of the magnetic from the astronomic meridian, for the (geographical) latitudes and longitudes of mid-Europe. The declination so obtained will rarely differ from the actual more than $\frac{1}{4}°$. This possibility of determining an astronomical direction with the magnetic needle is of the greatest value in surveys where only moderate accuracy is required.

On the use of the instruments concerned we will not touch further than to say that the universal directions for instruments for angular measurement are applicable to them. The accuracy is principally dependent on the length of the compass-needle, since the shorter it is the greater is the possible difference between its magnetic and geometrical axes.

The influence of friction on the point is lessened by slightly shaking the compass before reading. It is obvious that both ends of the needle should always be read.

58.—Measurement of Horizontal Intensity of the Earth's Magnetism.

This important measurement depends on two observations—viz. of a *time* of oscillation and of an *angle* of deflection. From the first may be obtained the product MT, of the horizontal intensity T of the earth's magnetism, and the magnetic moment M of the swinging magnet, if the moment of inertia of the magnet be known. The ratio $\frac{M}{T}$ is found by observing the deflection of another magnetic needle, caused by bringing the first to a measured distance from it. From these two numbers M may be eliminated by division and T determined.

It is to be noted that all times are given in seconds, lengths in millimetres, and masses in milligrammes.

I. *Determination of MT.*

The period of oscillation of the magnet, suspended by a thread, and swinging in a horizontal plane, is determined.

If, then,

t = the period of oscillation in seconds, and reduced to an infinitely small arc (51, 52);
K = the moment of inertia of the magnet (53);
Θ = the ratio of torsion of the thread (54);

the required product MT—

$$M \, . \, T = \frac{\pi K}{t^2 (1 + \Theta)} .$$

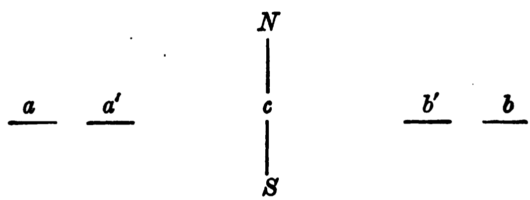
This number we will call A.

In the suspension of small bars cocoon silk is always employed on account of its feeble torsion; either in single fibres or in bundles of several threads. Such a bundle may be produced by fixing two glass rods on the edge of the table, as far apart as the required length of thread. The silk is

wound round them, its ends knotted, and the rods drawn slightly farther asunder, so as to stretch the thread, and the loop so formed suitably fixed to the magnet and point of suspension. A single cocoon fibre will sustain about 15 grm. without danger of breaking. Bars exceeding 1 pound in weight must be hung by wires (preferably of steel). Compare p. 134.

II. *Determination of* $\frac{M}{T}$.

The above bar magnet, of which we have called the magnetic moment M, is twice placed at equal distances to the east and west of a horizontal movable magnetic needle, and each time the angle is observed through which the latter is deflected from the magnetic meridian. From these observations the ratio of the magnetism of the needle to the horizontal intensity of that of the earth is determined by the following rules,—c is the centre of the compass:—



The line NS represents the magnetic meridian, *i.e.* the position which the free needle takes. The deflecting magnet is placed east or west of the compass-needle, so that its centre is in the positions a, a', b', b, successively; the distances of the centre of the magnet from that of the compass are equal in pairs, $ac = bc$, $a'c = b'c$.

The bar is placed, for instance, at a, with the north pole westward. The position of the compass-needle is read off at both ends. The deflecting bar is turned round 180°, so that its opposite pole is towards the compass-needle, which is now deflected in the opposite direction, and again read at both ends. The differences of the two positions of each end are halved, and the arithmetical mean of the two halves taken as the angle of deflection for the position a of the deflecting magnet.

It is supposed in the above that the circle of the compass is divided in one direction from 0° to 360°, the most convenient arrangement. If, as is sometimes the case, they are two zeros, from each of which it is numbered to both sides, instead of the half differences of the readings, we must, of course, take their half sum. The arithmetical means of the nearly equal angles for *ab*, and for *a′b′*, must then be taken (each will thus be the result of 8 single observations).

If we take

φ = the mean angle of deflection for a and b;
φ' = „ „ a' and b';
r = the half-length ab in millimetres;
r' = „ $a'b'$ „

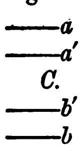
then the required number—

$$\frac{M}{T} = \frac{1}{2}\,\frac{r^5 \tan\varphi - r'^5 \tan\varphi'}{r^2 - r'^2}.$$

The number thus obtained, and which we denote by B, immediately gives the required intensity T, if we divide $MT = A$ by $\frac{M}{T} = B$, and extract the square root—

$$T = \sqrt{\frac{A}{B}}.$$

Proof for a Short Needle.—If a short needle, lying in a line with the magnetic axis of a bar magnet of magnetic moment M, and with poles east and west, and at a distance r from the centre of the needle, be deflected through the angle φ, then (compare Appendix) $\tan\varphi = \frac{2}{r^3}\frac{M}{T}\left(1 + \frac{a}{r^2}\right)$, where a is a constant for each magnet. If at a second distance r' the deflection φ' be observed, we have similarly $\tan\varphi' = \frac{1}{r'^3}\frac{M}{T}\left(1 + \frac{a}{r'^2}\right)$. By multiplying the first equation by r^5 and the second by r'^5, and by subtraction, the unknown constant a is eliminated, and we obtain $r^5 \tan\varphi - r'^5 \tan\varphi' = 2\frac{M}{T}(r^2 - r'^2)$. (See example below.) $\frac{M}{T}$ may also be obtained by plac-



ing the deflecting magnet, as in the annexed diagram, at equal distances successively *north* and *south* of the compass *C.*, and obtaining two pairs of such observations for different distances as before. The same mode of procedure must be followed as has been previously described, both in regard to the observation and in calculating the mean value. Using also the same notation as before for the distances of the centre of the deflecting magnet from the compass—viz. $r = \frac{1}{2}\, ab$, $r' = \frac{1}{2}\, ab'$; and further, taking φ and φ' for the mean angles of deflection for the positions ab and $a'b'$, it is only necessary to omit the factor $\frac{1}{2}$ from the previous result, and to take

$$\frac{M}{T} = \frac{r^5 \tan \varphi - r'^5 \tan \varphi'}{r^2 - r'^2}.$$

The method of observation above described achieves the following aims:—By reading the angle of deflection from both ends of the needle, and taking the mean, the influence of any eccentricity of its centre with regard to the graduated circle of the compass disappears. The poles of the deflecting magnet are reversed, to eliminate the effect of any unsymmetrical magnetisation in itself. The same thing is accomplished for the compass-needle by causing the deflections alternately to each side.

It is obvious that the exactness of the results will be increased in proportion to the eightfold repetition of each single reading. In order that the errors of observation may have the least possible influence on the result, it is best that the ratio of the two distances $r : r'$ should equal $4 : 3$. The angles of deflection should be as large as possible, but to produce this the deflecting bar must not be brought so near the needle as to make *the lesser distance* $a'b'$ *less than* 6 *times the length of the bar.* The length of the compass-needle should not be more than $\frac{1}{20}$th of $a'b'$.

Simplification by repeated employment of the same Magnet.—The deflection at two different distances is necessary for the elimination of the unknown distribution of magnetism in the bar and the needle, which is accomplished by the foregoing formula. If the same bar and needle be repeatedly used for the determination of T, the observation and calcula-

tion may be simplified. It is only necessary, once for all, to make the observation for two different distances. From this is calculated the factor X—

$$X = r^2\, r'^2\, \frac{r'^3 \tan \varphi' - r^3 \tan \varphi}{r^5 \tan \varphi - r'^5 \tan '}.$$

If, then, the angle of deviation Φ be found for one suitable distance R of the bar—

$$\frac{M}{T} = \frac{1}{2}\, \frac{R^3 \tan \Phi}{1 + \frac{X}{R^2}};$$

or, similarly, omitting the factor $\frac{1}{2}$, by the 2d method (*vide antea*).

If the deflection be measured with a magnetometer with mirror and scale, instead of with a compass, the procedure is the same as above described (except the reading from both ends of the needle). The scale-divisions must be reduced to angular measure by article 22. It is, however, necessary to take account of the torsion of the suspending thread by multiplying by $1 + \vartheta$; ϑ being the ratio of torsion (article 26) for the deflected bar—

$$\frac{M}{T} = \frac{r^5 \tan \varphi - r'^5 \tan \varphi'}{r^2 - r'^2}\,(1 + \vartheta).$$

The observations of oscillation and deflection should, of course, be made in the same place. Iron articles, which might exercise a local influence (and especially articles in the pocket of the observer, or steel spectacles) must be removed from the neighbourhood. Variations of magnetism of the earth or of the bar (the latter specially through change of temperature) are most likely to be excluded when the two sets of observations follow each other as closely as possible.

Example.—MEASUREMENT OF T WITH WEBER'S PORTABLE MAGNETOMETER.

1. DETERMINATION OF MT.

Moment of Inertia.—The magnetic bar is a right-angled parallelopiped, of which the length $a = 100$ mm., and the breadth $b = 12{\cdot}5$ mm. Its weight $m = 119860$ mgr. By 53 (p.126) its moment of inertia—

$$K = 19860 \frac{100^2 + 12{\cdot}5^2}{12} = 101440000.$$

Ratio of Torsion of Suspending Thread.—It was found that a single complete rotation of the thread produced a deflection of the magnet of 1°·4. By (14) the ratio of torsion—

$$\Theta = \frac{1{\cdot}4}{360 - 1{\cdot}4} = 0{\cdot}0039.$$

Time of Oscillation.—This was found to be (51) 7·414 sec., where the mean arc of oscillation was 30°. This, reduced to an infinitely small arc, gives for time of oscillation—

$$t = 7{\cdot}414 - 7{\cdot}4{\cdot}14 \times 0{\cdot}0043 = 7{\cdot}382 \text{ sec.}$$

Calculation of MT. The required value is—

$$MT \frac{\pi^2 K}{t^2 (1 + \Theta)} = \frac{3{\cdot}1416^2 \times 101440000}{7{\cdot}382^2 \times 1{\cdot}0039} = 18301000$$

2. Determination of $\frac{M}{T}$.

A compass stands on the 500th duration of a rule divided into millimetres, and perpendicular to the needle. The same magnet as was used in the determination of *MT* is placed with its centre successively on the 100th, 200th, 800th, and 900th divisions, twice on each division—once with its north and once with its south pole towards the compass. In these positions the following observations of the needle were made:—

When the magnet was placed on 100 the readings were—

	1st end.	2d end.
N pole towards needle	99°·4	279°·8
S pole towards needle	79°·9	260°·6
Half difference	9°·75	9°·60
Mean	9·675	

In a similar manner was found, when the centre of magnet lay—

at 200 mm.	22°·41
„ 800 „	22°·67
„ 900 „	9°·865

The two angles of deflection, φ and φ', obtained by taking the mean of each similar pair of observations, are as follows:—

$$\varphi = 9°{\cdot}77 = 9°46' \qquad \varphi = 22°{\cdot}54 = 22°32'$$

The two distances r and r' are—

$$r = \tfrac{1}{2}\,(900 - 100) = 400 \text{ mm.} \quad r' = \tfrac{1}{2}\,(800 - 200) = 300 \text{ mm.}$$

From these we now obtain (p. 137)—

$$\frac{M}{T} = \tfrac{1}{2}\,\frac{400^5 \tan 9^\circ 46' - 300^5 \tan 22^\circ 32'}{400^2 - 300^2} = 5387800.$$

The required horizontal intensity of terrestrial magnetism is therefore—

$$T = \sqrt{\frac{18301000}{5387800}} = 1{\cdot}843.$$

The factor X calculated for our magnet from this investigation will be—

$$X = 400^2 \times 300^2\,\frac{300^3 \tan 22^\circ 32' - 400^3 \tan 9^\circ 46'}{400^5 \tan 9^\circ 46' - 300^5 \tan 22^\circ 32'} = 3627.$$

In the case when only one observation of deflection $\varphi' = 22^\circ 32'$ for the distance 300 mm. has been made, the calculation by the formula (p. 139) gives—

$$\frac{M}{T} = \tfrac{1}{2}\,\frac{300^3 \tan 22^\circ 32'}{1 + \dfrac{3627}{300^2}} = 5387800,$$

the same value as above.

Note.—In the form of magnetometer used by the English government, a bar, turning on a horizontal graduated circle, carries the telescope, the deflecting magnet, and the point of suspension of a needle, furnished with a mark and collimating lens, as described in article 56. In observing, the telescope and bar are turned till the mark (cross-wires, scale) of the needle coincides with cross-wires of the telescope, the circle is read, the deflecting-bar placed in its carriage at a certain distance from the needle, and the whole turned till the mark again coincides, and the difference read as the deflection. The observations are conducted in other respects precisely as above, and the results reckoned by the same formulæ, except that $\sin \varphi$, etc., is substituted for $\tan \varphi$, etc. The method has the advantage of keeping the needle always in the same relative position to the deflecting bar, and of introducing no variation of torsion in its suspending fibre. The same apparatus is also adapted for measurement of declination, as the telescope is movable in altitude (see *Admiralty Manual of Scientific Inquiry*, p. 96 ; and Airy on *Magnetism*, p. 57). Under some circumstances, and where

very great accuracy is required, corrections must be introduced for change of temperature of the magnets, and for magnetism induced in them by that of the earth (see Airy, p. 165; *Ad. Manual*, p. 100).—*Trans.*

59.—Determination of Horizontal Intensity by the Compensated Magnetometer.

The compensated magnetometer is principally intended for the comparison of horizontal intensity at different places, and consists of a compass and a frame carrying four magnets of similar form to the compass-needle. The two smaller of these are twice the length, breadth, and thickness of the compass-needle, while the larger are threefold. When the frame is placed with its four holes on corresponding pins on the compass, the smaller bars deflect the needle from east and west (p. 136), and the larger ones from north and south (p. 138). The deflecting force of all the bars must act in the same direction, and therefore the poles of the smaller magnets must be in the opposite direction to those of the larger ones. The distance between the larger bars should be about 1·204 times that of the smaller ones.

Observation of Deflection.—The compass is so placed that when the frame is set upon it the line connecting the larger magnets is in the magnetic meridian. The frame being put on, the position of the needle is observed, the frame is turned 180° in its plane, and the position again noted, both ends of the needle being read each time. The half difference of these two positions is the angle of deflection.

Observation of Period of Oscillation.—A small pin is screwed into one of the holes near the large magnets, and by this the frame is hung in a stirrup attached to a cocoon thread. A mirror may also be screwed into another hole near the point of suspension for observation with telescope and scale. To determine the moment of inertia, two cylindrical weights are employed, which are hung by cocoon threads over the outer end-surfaces of the frame. (Compare also *Pogg. Ann.*, Bd. 142, S. 547.)

I. *Comparison of Horizontal Intensity at Two Places.*

If the magnetism of the deflecting bars may be considered unchanged between two observations (that is, when there is only a short time between them and the temperatures of the two places are nearly equal), it is only necessary to observe the angles of deflection, φ_1 and φ_2. The horizontal intensities of the places are inversely proportional to the tangents of these angles—

$$\frac{T_1}{T_2} = \frac{\tan \varphi_2}{\tan \varphi_1}.$$

If, however, the magnetism of the bars be altered, we must also observe the times of oscillation, t_1 and t_2, of the frame in the two places, when all four magnets have their poles in the same direction. Then

$$\frac{T_1}{T_2} = \frac{t_2}{t_1}\sqrt{\frac{\tan \varphi_2}{\tan \varphi_1}}.$$

II. *Determination of Absolute Horizontal Intensity.*

We take

$2r$ the distance of the centres of the smaller (east and west) magnets from each other;
$2R$ that of the larger magnets;
φ the angle of deflection;
t the time of oscillation with magnets all in the same direction;
τ that when the smaller magnets are turned 180°;
Θ the ratio of torsion of the thread in the first case;
K the moment of inertia.

We then have the absolute horizontal force—

$$T = \frac{\pi}{t\,\tau}\sqrt{\frac{K}{\tan \varphi}\left(\frac{\tau^2 - t^2}{r^3} + \frac{\tau^2(1-2\,\Theta) + t^2}{2R^3}\right)}.$$

The centre of a magnet is considered to be that of the pin on which it is movable.

To eliminate any want of symmetry of the magnet about this point, the angle of deflection may be twice observed—the

second time after turning all the magnets 180° on their pivots; the mean of the two angles being taken as φ.

With a deflecting magnet to the east or west of the needle (p. 136), the deflection of a short needle increases with diminished distance more rapidly than the inverse cube of the latter; with one acting from the north or south more slowly (p. 138); that is, the quantity denoted by a in the note (p. 137), is in the former case positive, and in the latter negative. These corrections compensate each other with similarly-formed magnets, when the dimensions are as 2 to 3, and the distances as 1 : 1·204 (see *Pogg. Ann.*, Bd. 142, S. 551). If also we denote by m and M the sum of the magnetic moments of the larger and smaller magnets respectively, in our instrument $\left(\frac{2m}{r^3}+\frac{M}{R^3}\right) \cos\varphi = T \sin\varphi$. The time of oscillation t, with similarly directed magnets, yields the result $(M+m)\,T = \frac{\pi^2 K}{t^2\,(1+\Theta)}$. Hence follows at once the formula under I., if the ratio $m : M$ be considered constant, and the ratio of torsion is small.

The time of oscillation, with oppositely directed magnets, yields $(M-m)\,T = \frac{\pi^2 K}{\tau^2\left(1+\Theta\,\frac{M+m}{M-m}\right)}$; and by combining these two equations the formula under II. follows by elimination of M and m; and lastly, by writing $1-2\,\Theta$ for $\frac{1-\Theta}{1+\Theta}$.

60.—Bifilar Magnetometer.

In order to measure the variation of horizontal force at the same place at different times, a magnet is hung by two threads equidistant from its centre, so that it lies horizontally. The line joining the two upper, and that connecting the two lower points of attachment of the threads, are turned till they form such an angle that the moment of rotation caused by terrestrial magnetism is balanced by that from the weight of the suspended magnet when the latter is at right angles to the magnetic meridian. It is best that this angle of torsion should be about 45°.

The slight rotation (read by mirror and scale) which is

then caused by variations in the horizontal intensity of terrestrial magnetism may be taken as proportional to this variation. Increasing intensity moves the north pole of the magnet towards the north; it is therefore convenient when motion in this direction corresponds to increasing numbers of the scale.

In order to determine the value of a scale-division in absolute measure, a horizontal magnet of known magnetic moment M (following article) is brought near the magnetometer, at the same height as the needle, and at a considerable measured distance r millimetres to the north or south. The reading of the magnetometer will differ n divisions on turning the north and south poles of the bar towards it. Then one division denotes a change of horizontal force of

$$\Delta = \frac{4M}{nr^3}.$$

When, as is customary, the value of the scale is to be expressed in fractional parts of the horizontal force of the place, it is only necessary to divide Δ by T (Table 22).

If, also, the scale-reading p corresponds to the horizontal force T, that of p' will be—

$$T' = T\left[1 + \frac{\Delta}{T}(p' - p)\right].$$

The bifilar magnetometer in this simple form is only adapted for the observation of variations of intensity in short spaces of time, since, with change of temperature, the distance between the suspending threads and their length is variable, and the magnetic moment of the bar may also alter with time.

Demonstration.—Taking m as the magnetic moment of the bifilar bar, the terrestrial magnetism will exert on it a moment of rotation mT. By a change in T of say Δ, the moment of rotation will vary Δm. The approach of the magnet M to the (great) distance r increases or diminishes the moment $\frac{2Mm}{r^3}$ (compare Appendix). If a deflection of n scale-divisions be thus produced, we have $\Delta : 1 = \frac{4M}{r^3} : n$.

61.—Determination of the Magnetism of a Bar in Absolute Measure.

The method described in article (58) is completely applicable to this case. It is only necessary to eliminate T from the two numbers $MT = A$ and $\frac{M}{T} = B$ by multiplication, and we obtain $M = \sqrt{AB}$. But M, as we have seen, is the magnetic moment of the bar employed for deflection and oscillation, expressed in Gauss's absolute measure. (Compare Appendix (4) on Absolute Magnetic Measure.

The magnet employed in the previous example has magnetic moment $M = \sqrt{18301000 \times 5387800} = 9929800$.

II. *Determination by Observations of Deflection.*

As the magnetism of bars varies with time and change of temperature, great exactness is seldom demanded, and since the horizontal intensity for the place of observation is approximately known (the value given in Table 22 being seldom more than 1 op. in error), the observations of deflection (58, II.) alone are sufficient.

In most cases it is enough to observe a single deflection at one distance. If we call

T the horizontal intensity of terrestrial magnetism;
r the distance of the centre of the magnet from that of the needle in millimetres;
φ the angle of deflection of the latter by the magnet;

the magnetic moment M of the magnet is given by the formula—

$$M = \tfrac{1}{2} r^3 \, T \tan \varphi,$$

if the deflecting magnet be east or west of the needle, as in the figure on p. 136; or

$$M = r^3 \, T \tan \varphi,$$

if it be north or south (p. 138). This formula is only irgorously exact when the lengths of the magnet and needle

are infinitely small compared to the distance between them. So long, however, as the distance r between the magnet and the needle is at least

3, 4, 5, 6, or 7 times the length of the magnet,

a short needle being also employed, the error introduced by the simplified determination cannot exceed at most

6, 3, 2, $1\frac{1}{2}$, or 1 op. of the whole value.

In this case the method of angular measurement, with mirror and scale, may be used with advantage, the torsion of the thread being corrected by multiplying T by $(1 + \Theta)$ (article 54).

In the examination of a magnet not in the form of a bar, as, for instance, a magnetic mineral, the magnetic axis of which cannot be determined from its form, the body is turned into that position in which it produces the greatest deflection. By this means we obtain at the same time the position of the magnetic axis—viz. in the "first position" (p. 136), as the line joining the centres of the magnet and needle; and in the "second position" (p. 138), as the perpendicular to this line. Instead of this we may determine the components of the magnetic moment in three positions perpendicular to each other. If we obtain for these the values M_1, M_2, M_3, we have $M = \sqrt{M_1^2 + M_2^2 + M_3^2}$. The position of the magnetic axis may also be found, since $\frac{M_1}{M}$, $\frac{M_2}{M}$, and $\frac{M_3}{M}$, are the cosines of the angles which it makes with the three experimental directions.

III. *Determination by Oscillation.*

In a bar of regular form we may easily calculate the moment of inertia K (53), and then we have from the time of oscillation t, neglecting the torsion of the suspending thread—

$$M = \frac{{}^2K}{t^2T}.$$

The torsion may be eliminated by employment of a

stirrup of such weight that its time of oscillation alone is the same as that with the magnet upon it.

The number obtained by dividing the magnetic moment by the weight of the magnetic body in milligrammes is called its specific magnetism. In the best magnets of elongated form this may amount to about 1000.

62.—Ohm's Laws of Galvanic Currents.

I. *In simple undivided Circuits.*

1. The electrical resistance w of a cylindrical conductor is directly proportional to its length l, and inversely to its sectional area q; or $w = k\frac{l}{q}$. The factor k varies in value in different substances, and is called the *specific resistance* of the body. As we ordinarily take $\frac{1}{w}$ as the *conductivity;* so we may call $\frac{1}{k}$ the *specific conductivity* of a conductor.

If the Siemens or mercury unit of resistance (resistance of a column of mercury 1 metre in length, and 1 $\square^{\text{mm.}}$ section) be employed, the specific resistance of mercury at 0°c must be taken as unity. The resistance of a cylindrical body of length l and section q $\square^{\text{mm.}}$, is then expressed in mercury units by the number $k\frac{l}{q}$, k being the specific resistance referred to quicksilver. Conversely, if we find that in a cylindrical conductor (wire, column of fluid in a prismatic vessel), of length l, and sectional area q $\square^{\text{mm.}}$, the resistance $= w$ Siem., the specific resistance of the material $k = w\frac{q}{l}$, and its specific conductivity $\frac{1}{k} = \frac{l}{w.q.}$ referred to quicksilver.

The specific conductivity of the most important substances is given in Table 24.

2. The total resistance of a circuit is the sum of the resistances of all the separate parts.

3. The total electromotive force is similarly the algebraical sum of all the separate electromotive forces.

4. The current strength or intensity i is directly proportional to the electromotive force e, and inversely so to the resistance w, or—

$$i = C\frac{e}{w}.$$

The numerical value of the factor C depends on the units in which i, e, and w are measured. It is most simple when these are chosen as to $C = 1$. For example, we have such a system of galvanic units if we express the *current strength* in *magnetic measure* (66, p. 154), the *resistance* in *Siemens's units,* and *electromotive force* in units of which that of a Grove's cell = 20·0, or of a Daniell's = 11·7. We have then simply $i = \frac{e}{w}$. For example, a battery of 8 Grove's cells produces in a closed circuit of 100 Siems. resistance a current of $\frac{8 \times 20}{100} = 1{\cdot}6$ magnetic units. (Compare also Appendix 5 on Absolute Galvanic Measure.

II. *Derived Currents in divided Circuits.*

If a current between two points of the undivided conductor branches into several paths of the resistances w_1, w_2 . . . . and in which correspondingly we have the currents i_1 i_2 . . . . . .

5. The sum of the divided currents is equal to the undivided current, or $i_1 + i_2 \ldots\ldots = i$.

6. The divided currents are inversely proportional to the relative resistances of their respective paths (or directly to their conductivities, $i_1 : i_2 : \ldots = \frac{1}{w_1} : \frac{1}{w_2} : \ldots\ldots$

7. The total conductivity of the divided circuit is the sum of all the conductivities of the single branches : $\frac{1}{w} = \frac{1}{w_1} + \frac{1}{w_2} + \ldots\ldots$

The laws given above, from 2 to 7, are combined by Kirchoff in the two following, which give directly the equations for currents in divided conductors.

I. At any point of division, the sum of the current-strengths in all the branches $= 0$, taking the currents *towards* the juncture as of opposite sign to those *from* it.

II. If we consider any part of the conductor which forms a closed circuit in itself, and reckon in it all the electromotive forces and currents in one direction as positive, and in the other negative, the sum of the products of the individual resistances into the current-strengths is equal to the sum of the electromotive forces.

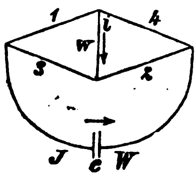


Fig. 11.

For example, in Wheatstone's bridge we obtain six equations by taking the divided currents and their corresponding resistances—

$$
\begin{array}{lcl}
I - i_1 - i_2 = 0 & \qquad & IW + i_1 w_1 + i_4 w_4 = E \\
I - i_3 - i_4 = 0 & & iw \;\; - i_1 w_1 + i_3 w_3 = 0 \\
i + i_1 - i_4 = 0 & & iw \;\; - i_2 w_2 + i_4 w_4 = 0
\end{array}
$$

The remaining equations (as $i + i_2 - i_3 = 0$) are contained in the above.

63.—Measurement of Currents with the Tangent-Galvanometer.

For many purposes relative measurement, or determination of the ratio of current-strengths, only is sufficient. We will therefore consider this case first.

The tangent-compass, or tangent-galvanometer, consists of a multiplier fixed with the plane of its coils in the magnetic meridian. In the centre is a compass, of which the needle must be very short compared to the diameter of the coils.

If two currents passed through the multiplier deflect the needle respectively φ and φ', their relative strengths (intensities) are proportional to the tangents of the angles of deflection, or—

$$i : i' = tan\ \varphi : tan\ \varphi'.$$

For accurate measurement the angles of deflection should

be neither very large nor very small, those of about 45° being most advantageous. It is necessary, therefore, for currents of very different intensities, to employ galvanometers of different degrees of sensitiveness; that is, with coils of different diameters or of different lengths, or the instrument may be so constructed that the current may be passed through a greater or less number of coils as required. The results of different instruments may be compared with each other by passing the same current through both at once. If, for instance, we obtain in this manner a deflection of 66°·5 in the first instrument, and of 14°·2 in the second, the tangents of the angles of deflection of No. 1 must be multiplied by $\frac{tan\ 14^\circ{\cdot}2}{tan\ 66^\circ{\cdot}5} = \frac{0{\cdot}253}{2{\cdot}30} = 0{\cdot}110$, to make them comparable with those of No. 2. The method of finding the reduction-factor from the number and dimensions of the coils is given in 66 (p. 154).

Commutator.—The tangent-compass is usually adjusted so that the needle points to zero when the plane of the coils is in the magnetic meridian. Whether this is accurately the case, must be tested, preferably by the employment of a very short needle, for the proportionality of current-strengths to the tangents of the angles of deflection only holds good if the instrument be exactly placed, and especially so with powerful currents. This difficulty may, however, be easily avoided by passing the current successively in opposite directions through the galvanometer, and taking the mean of the deflection to both sides (half the combined deflection) as φ. In the value thus obtained, errors from incorrect position are eliminated. It is convenient for this purpose that a commutator should be permanently connected with the galvanometer, which will allow the current to be reversed without altering any other part of the circuit. This gives the additional advantage of a double degree of accuracy, and renders it unnecessary to observe the zero point exactly; and lastly, a well-arranged commutator serves conveniently to open and close the circuit.

Deviation from the Law of Proportionality of Tangents.—In order that the proportionality of tangents to current-strengths may be accurate within 1 op. for all angles, the length of the needle must not at most exceed $\frac{1}{12}$ the diameter of the coils. A short needle, with a long attached pointer (for instance a thread of glass, attached with varnish, or a slip of aluminium foil creased down the centre to give it stiffness), should be employed (compare also p. 156). The deviation from the law of tangents may also be diminished by placing the needle not in the plane of the coils, but about $\frac{1}{4}$ their diameter to one side of it. An error of 1 op. would then only be produced by a needle of $\frac{1}{4}$ a diameter in length, while with one not exceeding $\frac{1}{8}$ the error may be considered inappreciable (Gaugain, Helmholtz).

For reading the position of the needle at rest, or when only slightly deflected, two pointers at right angles to the needle are convenient. To avoid parallax in reading, the compass may be furnished with a piece of looking-glass in the centre, above which the eye must be held so that its reflected image coincides with the needle or index. In exact measurements both ends of the needle should always be read (compare p. 136).

To bring the needle to rest, a small magnet may be employed, which is brought near or withdrawn as required. The commutator may also, with practice, be employed for the same purpose; in particular by reversal of the current, the circuit being at first merely interrupted, and only closed at the instant when the needle begins to return from an oscillation to the opposite side.

64.—Sine-Galvanometer or Sine-Compass.

The sine-galvanometer, like the tangent-galvanometer, consists of a multiplier and a firmly-attached compass, or, in place of the latter, a needle with a single position-mark. The multiplier is itself movable over a second graduated circle.

In each measurement with the sine-compass the coils are

turned on this second circle till they make the same angle with the needle (compass-angle) as before; or, in other words, the needle is always brought to the same division of its circle.

The strength of the current is then proportional to the sine (Table 30) of the angle of deflection φ—viz. the angle through which the multiplier has been turned to make the compass-angle the same as it was without current—

$$i : i' = sin\ \varphi : sin\ \varphi'.$$

In the measurement of feeble currents small compass-angles must be employed; in that of powerful currents the angles must also be large. Results so obtained are not directly comparable, but a factor may easily be determined, multiplication by which will reduce the measures of one compass-angle to those of another. For this purpose the same current must be observed with the two compass-angles, 1 and 2, to be compared, and the corresponding deflections noted. Then $n = \frac{sin\ a_1}{sin\ a_2}$, by which all measurements with compass-angle 2 must be multiplied to reduce them to the same value as those with 1. We may thus reduce the compass-angles 0°, 30°, 45°, 60°, 70°, to the same measure.

The advantage of the sine-compass is, that the law of sines is independent of the size and form of both needle and multiplier; the disadvantages are its troublesome adjustment, and the doubled sources of error in reading. The limit of difference of current-strengths which can be compared by this instrument is the same as that of the tangent-compass if the coils are of large diameter, and wider if they be narrow.

65.—Mirror Galvanometers

Fixed multipliers, which closely surround the needle, can only be employed in general as galvanoscopes, or to test which of the currents compared is the greater. If, however, very small deflections only are observed with the mirror and scale (47, 48), the currents are proportional to the tangents

of the angles of deflection; or, if the latter do not exceed a few degrees, to the angles themselves measured in divisions of the scale. The limit within which this is allowable is naturally dependent on the dimensions of the multiplier and needle.

A simple method for determining the factor to reduce such readings to absolute measure will be found at the end of (68).

Should a galvanometer with narrow coils be employed to measure currents producing considerable deflections, the only way is to graduate it empirically by comparison with the sine or tangent galvanometer, or voltameter (67), or with currents of known quantity. From the deflections so obtained a table may be constructed (graphically or otherwise) for interpolation of intermediate readings.

66.—Measurement of Currents in Absolute Magnetic Units with the Tangent-Compass.

The methods previously described only yield comparable results when the observations are made with the same instrument, since in each case we employ a merely arbitrary unit dependent on the dimensions of the instrument and the horizontal intensity of terrestrial magnetism. In order to reduce measurements to a universally intelligible unit we may employ the magnetic or Weber's* unit of current-strength, which may be defined as that current which exerts a unit of magnetic force (compare Appendix, A, 5, on Magnetic Current-Measure). From a measurement by the tangent-compass we may calculate the value of the current in this unit in the following manner. Calling

n the number of the circular windings of the multiplier;
r their mean radius in millimetres;
T the horizontal intensity of terrestrial magnetism (58 and Table 22);
the angle of deflection of the needle;

* Weber's unit $= 1 \dfrac{\text{mm.}^{\frac{1}{2}}\ \text{mgr.}^{\frac{1}{2}}}{\text{sec.}}$.

then the strength i of the current which produces this deflection is, in magnetic measure—

$$i = \frac{rT}{2n\pi} \tan \alpha ,$$

and we may call $\frac{rT}{2n\pi}$ the reduction-factor of magnetic measure.

Proof.—The total length of the coils is $2nr\pi$. The current i tends to turn the needle perpendicularly to the plane of the coils, and exerts on a short needle in the centre of the coils, and deflected through the angle α from their plane, a moment of rotation $2nr\pi \frac{iM}{r^2} \cos \alpha$. α is also the deflection from the magnetic meridian, and the earth's magnetic force produces a moment of rotation, $MT \sin \alpha$, in the contrary direction. The formula is obtained by combining these results.

The mean radius r of the coils is most easily obtained, for multipliers with many windings, by dividing the total length l of the wire by the number n of the coils multiplied by 2π, or—

$$r = \frac{l}{2n\pi}.$$

The reduction-factor of a tangent-compass will, of course, vary with time and place, since it is dependent on the intensity of terrestrial magnetism. For instance—

at the beginning of	1870.	1875.	1880.
in London	T = 1·78	1·80	1·82
„ Göttingen	= 1·850	1·860	1·888
„ Darmstadt	= 1·91	1·93	1·95
„ Zürich	= 2·00	2·02	2·04

For places where T has not been determined it may be taken from Table 22—all local influences, such as iron objects, and especially long iron conductors, being as much as possible removed from the neighbourhood.

Care must be taken that the currents in the external conducting wires connected with the instrument do not affect the needle. This is most certainly accomplished by placing those leading to and from the instrument close beside each other.

Example.—A multiplier is formed by winding a wire 19480 mm. long, in 24 circular coils. Then $r = \frac{19480}{48 \times 3{\cdot}1416} = 129{\cdot}2$ mm. Taking $T = 1{\cdot}92$, the strength of a current which produces the deflection α is, in magnetic measure—

$$= \frac{129{\cdot}2 \times 1{\cdot}92}{2 \times 24 \times 3{\cdot}1416} \tan \alpha = 1{\cdot}645 \tan \alpha.$$

Formulæ of Correction for Length of Needle and Section of Coils.—It is assumed in the use of the above formula that the section of the coil is very small compared with its diameter. It frequently happens that this condition is imperfectly fulfilled in coils of many windings. If, as is generally the case, the coil is of rectangular section, the original formula may be corrected by writing, instead of $\frac{rT}{2n\pi}$, $\frac{rT}{2n\pi}\left(1 + \frac{1}{2}\frac{a^2}{r^2} - \frac{1}{3}\frac{b^2}{r^2}\right)$, where a is half the breadth and b half the depth of the rectangular section.

Lastly, if the length of the needle be not very small compared to the diameter of the coil, we must add to the above expression the factor $\left(1 - \frac{3}{4}\frac{l^2}{r^2}\right)$, and, instead of $\tan \varphi$, must write $\left(1 + \frac{15}{4}\frac{l^2}{r^2}\sin^2 \varphi\right) \tan \varphi$. Here l is half the distance between the north and south poles (the "centres of gravity of northern and southern magnetism"). If l be not determined by experiment, 0·85 of the whole length may be taken in an ordinary needle as the distance between the poles.

The complete formula, assuming that the corrections are small, will be—

$$i = \frac{rT}{2n\pi}\left(1 + \frac{1}{2}\frac{a^2}{r^2} - \frac{1}{3}\frac{b^2}{r^2} - \frac{3}{4}\frac{l^2}{r^2}\right)\left(1 + \frac{15}{4}\frac{l^2}{r^2}\sin^2 \varphi\right) \tan \varphi.$$

If the needle be suspended by a thread, of which the ratio of torsion $= \Theta$ (54), we must write $T(1 + \Theta)$ instead of T.

The remarks on the use of commutators and on reading both ends of the needle (pp. 151, 152) are, of course, applicable to the above.

67.—Current Measurement in Chemical Units with the Voltameter.

If the products of chemical decomposition produced by a current be measured by a voltameter, they always bear an exactly defined relation to the current-strength, and form a measure comparable with the magnetic by the aid of the following rules:—

1. The decomposition in a given time by different currents is proportional to the current-strength.

2. The decomposition products of the same current in different electrolytes are chemically equivalent. (Faraday's law.)

3. A current of unit strength in magnetic measure destroys 0·560 mgr. of water per minute. (As electro-chemical equivalent of water, Weber takes the current-strength 107, which decomposes 1 mgr. of water per second.)

As electrolytes are employed either water acidulated with sulphuric acid between platinum electrodes, or an aqueous solution of cupric sulphate or argentic nitrate, with copper and silver electrodes respectively. The dilute sulphuric acid should be chemically pure, and is best of the specific gravity of about 1·23. The metallic solutions may be prepared by diluting the saturated solution with an equal volume of water.

In current measurement with the voltameter, the current is passed for a measured time, and the products of decomposition determined. The amount of the latter divided by the time gives the quantity decomposed in unit of time. We will count the latter in minutes.

Volume Voltameter.—In the water voltameter the mixture of oxygen and hydrogen liberated is collected and measured in a divided glass tube. For the sake of exactness the

volume of the gases must be reduced to 0° C. and 760 mm. (18) by the formula—

$$v_0 = \frac{v}{1 + 0{\cdot}003665\,t} \cdot \frac{H}{760}.$$

Here

v is the observed volume ;
v_0 the volume reduced to 0° temperature and 760 mm. pressure ;
t the temperature of the observed gases ;
H the pressure in millimetres of mercury under which the gas was confined.

The liberated gas is almost always collected over a fluid. In this case, to find the pressure H, we take the height h of the fluid in the tube above the free surface Δ, the density of the fluid, and b the height of the barometer; then—

$$H = b - h\,\frac{\Delta}{13{\cdot}6};$$

13·6 being the specific gravity of mercury. If the gas be retained over mercury we have, of course, only $H = b - h$. With feeble currents the hydrogen only should be collected, and the volume of mixed gases calculated by multiplication by $\frac{3}{2}$, the oxygen being partially absorbed in the form of ozone. On the same account, it is advisable repeatedly to employ the same sulphuric acid. (See example, p. 159.)

If the gas be collected over the acidulated water itself, it may be regarded as saturated with aqueous vapour. To reduce it to its dry volume the tension e of aqueous vapour for the observed temperature (Table 13) must be subtracted from H. For the ordinary temperatures of rooms it is sufficiently exact to write, instead of e in mm., t in degrees Centigrade.

Weight Voltameter.—Instead of measuring the gas, the weight of the decomposed water may be determined by weighing before and after the experiment—a small drying apparatus, containing concentrated sulphuric acid, being attached to prevent escape of aqueous vapour with the disengaged gases. As the density of the mixed gases at 0°

and 760 mm. is 0·0005363, 1 cubic centimetre corresponds to 0·5363 mgr. of water. For approximate reduction we may note that under ordinary conditions (exactly at 16° C. and 750 mm.) 1 c.c. of the mixed gases weighs $\frac{1}{2}$ mgr.

In the copper and silver voltameters the current-strength is found by determination of the gain of weight of the negative electrode.

Reduction of the different Current-Measures to each other.—In the employment of different voltameters we have four separate definitions for the unit of strength of a galvanic current, viz.—

1. The volume of mixed gases at 0° and 760 mm. liberated in 1 minute. (In ordinary use—*Jacobi.*)
2. The weight of water decomposed in 1 minute
3. The weight of copper deposited in 1 minute.
4. The weight of silver deposited in 1 minute.

That we may have convenient numerical values, the volumes are reckoned in cubic centimetres, and the weights in milligrammes. To these chemical units we may add the magnetic unit (66) measured with the tangent-compass.

Very frequently it is necessary to reduce measurements of current-strength, obtained in one of these units, to those of another. For this purpose the information given above respecting the density of the mixed gases, and the quantity of water decomposed by a magnetic unit current will suffice, if the equivalent weights 9, 31·7, and 107·9, for water, copper, and silver respectively, be taken into account. For greater convenience, however, Table 25 gives reduction-factors to convert each measure into any of the others.

It is not uninteresting to note that by reducing the volume of the mixed gases to 800 mm., instead of the ordinary 760 mm., we pass from Jacobi's chemical to Weber's magnetic unit with considerable exactness, and quite sufficient for ordinary purposes.

Example.—Measurement of Current-Strength with the Volumetric Water-Voltameter.

The duration of the current was = 10 min., the volume of

liberated hydrogen = 18·4 c.c., the temp. = 14°, the height of barometer = 762 mm., and the gas was collected over a column of dilute sulphuric acid of the specific gravity 1·23, and, at the close of the experiment, 55 mm. high.

18·4 c.c. hydrogen correspond to 27·6 c.c. of mixed gases. The pressure of the gases is, as above, $H = 762 - 55\,\frac{1\cdot23}{13\cdot6} - 12 =$ 745 mm. It would, therefore, at 0° and 760 mm. (p. 158) have the volume $\frac{27\cdot6}{1 + 0\cdot003665 \times 14}\cdot\frac{745}{760} = 25\cdot74$ c.c. Consequently we have 2·574 c.c. of mixed gases liberated per minute, and this by Table 25—

$= 2\cdot574 \times 0\cdot5363 = \ 1\cdot380$ mgr. of water per minute.
$= 2\cdot574 \times 1\cdot889 \ = \ 4\cdot861$ „ copper „
$= 2\cdot574 \times 6\cdot432 \ = 16\cdot55$ „ silver „
$= 2\cdot574 \times 0\cdot9579 = \ 2\cdot465$ electromagnetic units.

68.—Determination of the Reduction-Factor of a Galvanometer.

If the number of windings of a galvanometer-coil be unknown, or if from its irregular shape or other cause the reduction-factor C cannot be calculated, it will be necessary to determine it experimentally. For shortness' sake, we will speak only of the tangent-compass, but if a sine-compass be employed, it is only necessary to substitute *sin* α for *tan* α.

I. *A Tangent-compass of known Reduction-factor* C' is connected in the same circuit with the instrument to be examined. If the deflections of the two instruments are respectively α' and α, then

$$C = C'\,\frac{\tan\alpha'}{\tan\alpha}.$$

II. *With the Voltameter.*—The same current is passed at once through both instruments for a measured time. If, then,

τ = the time;
m = amount of electrolyte decomposed in the voltameter;
α = the angle of deflection of the galvanometer;

the required reduction-factor C, or, in other words, the number by which the tangent of the angle of deflection must be multiplied to reduce it to absolute measure is—

$$C = \frac{m}{r \tan \alpha}.$$

Here C will be the reduction-factor for the special chemical measure of the voltameter employed, but by the aid of Table 25 it may easily be reduced to any other measure.

Since a current rarely remains constant for long together, and especially so with an intercalated voltameter, the position of the needle must be observed at regular intervals, say from minute to minute, during the experiment, and at the end the arithmetical mean must be taken as the true deflection. A commutator may be employed with advantage. The measurement will be most exact when the angle of deflection is about 45°.

As example, we will suppose that the current measured by the voltameter in the previous article gives a deflection of 42°·6. In this case the reduction-factor $C = \frac{25{\cdot}74}{10 \tan 42^\circ{\cdot}6} = \frac{2{\cdot}574}{0{\cdot}9195} = 2{\cdot}799$; and a current which, with this tangent-compass, produces the deflection φ, will liberate $2{\cdot}799 \tan \varphi$ cubic cm. per minute of mixed gases at 0° C and 760 mm. In magnetic measure the factor $= 2{\cdot}799 \times 0{\cdot}9579 = 2{\cdot}681$.

III. *By means of a known Electromotive Force.*—A very simple and universally applicable method follows from the law that current-strength is directly proportional to electromotive force and inversely to resistance, or that $i = \frac{e}{w}$, i being current-strength, e electromotive force, and w resistance. If the galvanometer be included in a circuit with n cells of electromotive force e, and if the deflection produced be α, and the total resistance (of cell, galvanometer, and rheostat) be w—

$$C = \frac{n\, e}{w \tan \alpha}.$$

The electromotive force of a Grove's or Bunsen's cell is

about 1·92, and that of a Daniell's 1·08, in absolute electromagnetic units. (See Appendix A, 5.) C will, of course, give absolute magnetic measure.

The method is specially applicable to reflecting galvanometers, and as the intercalated resistance in this case will usually be very large compared to the internal resistance of the cell and that of the galvanometer, these may usually be neglected, and w taken as equal to the rheostat resistance alone.

69.—Determination of Galvanic Resistance with the Rheostat.

This problem may be divided into two cases—viz. the proof of equality between like resistances, and the determination of the ratio of the amounts of unequal ones. When, by means of a rheostat (set of resistance-coils), we can increase one of the resistances at will by known amounts, we may always employ the method of equality. We will first consider this simplest case.

It is obvious that the same methods are adapted for copying resistances.

I. *Measurement of Resistance by Substitution* is dependent on the fact that two resistances must be equal, which, when substituted for each other in the same circuit, give the same current-strength. A circuit is formed, consisting of a galvanic cell E, a galvanometer G, and the rheostat R. The resistance, W, to be measured is shown in the figure as intercalated, but may be excluded by restoring a connection without sensible resistance at a. First, the position of the galvanometer-needle must be noted when W, and a sufficient length of rheostat wire to reduce the deflection to a convenient amount is included in the circuit. W must then

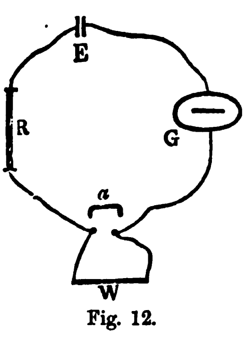


Fig. 12.

be excluded, and an amount of rheostat resistance added in its place sufficient to bring back the needle to its original position.

This added rheostat resistance is equal to the required resistance W.

If the resistance of the rheostat cannot be altered by sufficiently small intervals, but—as, for instance, in Siemens's resistance-coils, with plug arrangement—can only be altered by jumps, we must make use of a method of interpolation similar to that described in (7). The position of the needle is observed with the nearest resistances above and below that required, and if the difference of deflection is small, the increase of resistance may be taken as proportion to the decrease of current. If the observed position of the needle be

α with the required resistance w;
α_1 „ rheostat resistance w_1;
α_2 „ „ „ w_2.

Then

$$W = w_1 + (w_2 - w_1)\frac{\alpha_1 - \alpha}{\alpha_1 - \alpha_2}.$$

For accuracy and quickness this method of interpolation is always to be preferred.

Example.—

Intercalated	W	Rh. 14	Rh. 15
Deflection	$45°\cdot3$	$47°\cdot9$	$44°\cdot5$

$$W = 14 + \frac{47\cdot9 - 45\cdot3}{47\cdot9 - 44\cdot5} = 14\cdot76.$$

The method of substitution is almost universally applicable if the resistances are not too small, and it requires only a galvanoscope to prove the equality of the two currents. A *constant* element is, however, necessary. Any slight change in the latter is eliminated by repeating the observation and taking the mean, and is also diminished by rapid observations. For this reason it is best to make a rough measurement of W before the final determination.

II. *By simple Division of the Current through a Differential Galvanometer*, according to the law that the resistances of two conductors are equal, when, if inserted as two branches of a circuit, the current divides itself equally between them. The equality of the two currents is determined by the differential galvanometer, the coil of which consists of two wires of equal length wound together. One current passes through one wire, and the other through the other, in opposite directions; and thus, when the currents are equal, they neutralise each other's influence on the sensitive needle inside. The currents, therefore, are known to be equal when the needle is undeflected.

The annexed figure shows the connections for measurement of resistances. *G* represents the two coils of the differential galvanometer, with their ends brought out. The current of the cell *E* divides between the two middle ends, and so passes through the coils in opposite directions. From the outer ends one-half of the divided current is led through *W*, the resistance to be measured, and the other through the rheostat *R*, uniting again at the opposite pole of the cell. The amount of rheostat resistance intercalated to bring the needle back to its normal position is equal to the resistance *W*. The method of interpolation may be employed here.

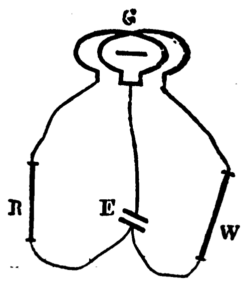


Fig. 13.

Testing the Differential Galvanometer.—In this method the differential galvanometer is assumed to possess two properties—firstly, that the current-strengths are equal when the needle is uninfluenced; secondly, that the resistances of the two coils are equal. The former is tested by passing the same current through both coils in opposite directions; that is, counting the terminals of the galvanometer from left to right, 1 and 2 must be connected with each other, and 3 and 4 each with a pole of the battery. The needle should remain undeflected. This being proved, the second require-

ment may be tested by allowing the current from one battery to divide itself through the coils without any further resistance, when the needle must again be undeflected. Any required correction of the instrument should be made in the above order.

Lastly, we may be independent of the exact fulfilment of these conditions if we connect W and R with a commutator, so that their positions may be easily reversed. *W and R are equal when reversal of their position does not influence the deflection of the needle.*

The connecting wires should be of feeble resistance, and of equal length and thickness for both divisions of the current.

The advantages of the method are its sensitiveness and independence of the element.

III. *By double Division of the Current with Wheatstone's Bridge.*—By the law, that in the arrangement shown in the figure the current-strength in G will be null when

$$a : b = R : W.$$

a and b are two resistances of equal amount, R is the rheostat, W the resistance to be measured, and G is a galvanoscope. W is equal to the rheostat-resistance, which must be inserted in order that G may show no deflection.

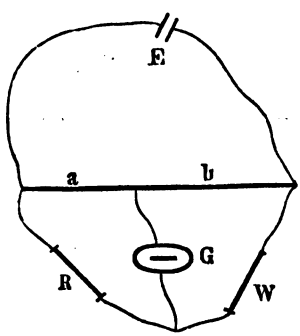


Fig. 14.

By the method described in II. we may be independent of the exact equality of a and b; the resistances R and W being equal when reversal of their positions leaves the needle undeflected.

The arrangement of the resistances may be changed, resistances of known equality being placed in a and R, and the resistances to be compared in b and W.

By reversing the positions of either pair of resistances

we may again be independent of the exact equality of a and b; the resistances W and R being equal when, on changing their places, the galvanoscope-needle is undeflected. This is most conveniently done by a suitable commutator.

As to advantages of methods of interpolation, and choice of connections, the remarks under II. are completely applicable.

70.—Comparison of Unequal Resistances.

Here we have the problem of measuring a resistance without a rheostat, but only with the unit in which the measurement is to be made.

I. *With the Galvanometer (Tangent, Sine, or Reflecting Galvanometer).*—A circuit is made, including the galvanometer and a constant cell, and the current-strength is measured. Let it be i. One of the resistances, W, is next included, and the current-strength again measured is i_1. The other resistance, W_2, is substituted for W_1, and gives a current-strength of i_2. The required ratio of the resistances may then be calculated—

$$\frac{W_1}{W_2} = \frac{i - i_1}{i - i_2} \cdot \frac{i_2}{i_1}.$$

For i, i_1, i_2, we naturally take the tangents or sines respectively of the angles of deflection. The method rarely gives exact results, as the electromotive force of almost all elements is dependent on the current-strength. Further, it involves of necessity all the difficulties dependent on current-measurement (63, 64), It is the less exact the smaller the resistances to be compared and the greater their difference. The formula follows from (62, 6), since $i : i_1 : i_2 = \frac{1}{w} : \frac{1}{w + w} : \frac{1}{w + w_2}$, where w is the resistance and i the current-strength.

Example.—The observed deflections of a tangent-compass were, including—

No resistance	$66°\cdot 8$	tangent = 2·333
The resistance to be measured	$23°\cdot 9$	tangent = 0·443
One Ohm	$46°\cdot 3$	tangent = 0·952

$$\text{Therefore } W_1 = \frac{2\cdot 333 - 0\cdot 443}{2\cdot 333 - 0\cdot 952} \times \frac{0\cdot 952}{0\cdot 443} = 2\cdot 94 \text{ Ohms.}$$

II. *Method with Wheatstone's Bridge.*—In the figure a and b represent two resistances, of which the ratio can be easily varied. This is the case when a and b together consist of a stretched (platinum) wire of uniform diameter, in which we may take the resistance as proportional to the length. On the wire is a movable contact (platinum), to which the connecting wire of the galvanoscope is attached. The same conditions are fulfilled when

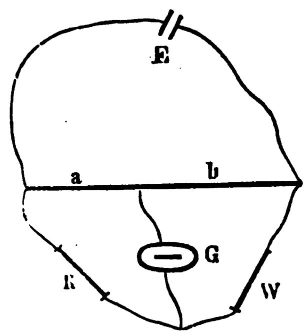


Fig. 15.

b is a rheostat and a a known resistance in rheostat units. W and R are the two resistances to be compared, and when no current passes through the galvanoscope, G, they bear the same relation as a and b.

$$\text{Therefore } \frac{W}{R} = \frac{b}{a}.$$

The connecting wires of R and W have no influence when their resistances are in the same ratio as $R : W$. Hence it is advisable roughly to determine this ratio by a preliminary experiment, and to approximate to it that of the lengths of wire (of the same sort) on each side. For this purpose it is convenient to join R to W by a single wire, and to connect the galvanometer wire to it by means of a movable binding screw.

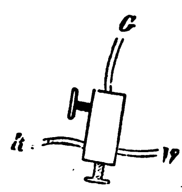


Fig. 16.

The positions of b and R may be reversed, as in No. III. of the preceding article.

Note.—It is desirable that the current should pass through the resistances for as short a time as possible, since by its prolonged passage they may become heated, and resistance varies with temperature. Sudden currents, however, are apt to produce induced currents in the coils, which pass through and disturb the galvanoscope. It is therefore advantageous to close first the battery circuit, and then that of the galvanoscope. This is most readily effected

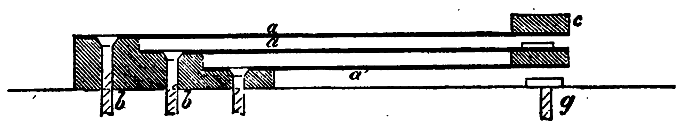


Fig. 16a.

by the simple contact-key described in the *Brit. Assoc. Rep.*, 1864, p. 353. This consists of three strips of thin brass *a a*, and *a'*; *a a* are connected with the battery and bridge at *b b*, and are brought in contact by pressing the small block of gutta percha *c*; while *a'*, also separated from *a* by a little block of gutta percha, connects the sliding contact with the galvanoscope by touching *g*.

It is desirable to determine the relation of the resistances roughly by an ordinary galvanoscope, before employing a very sensitive instrument, both to save time and to avoid injury to the more delicate one. These remarks of course apply to both the previous articles. See also *Brit. Assoc. Reports*, 1863, 1864, and 1867.—(*Trans.*)

III. *From the Damping of a Swinging Magnetic Needle* (50).—A needle swinging inside a closed coil induces currents in it by its motion, which react upon the needle in opposition to that motion. The diminution of arc of oscillation so caused is dependent not only on the needle itself, and on the form and number of the windings of the coil, but on the collective resistance, $w_0 + w$, of these, and of the wire closing the circuit. Theory shows that the logarithmic decrement (50) is inversely proportional to $w_0 + w$. w_1 and w_2 are the resistances to be compared. The logarithmic decrement λ_0 is observed when the multiplier, of which the resistance is w_0, is closed by a wire of no sensible resistance—

λ_1 when the resistance w_1 is included;
λ_2 when w_2 is substituted for w_1;
λ' with the open multiplier, and through the mechanical resistance of the air;

then—

$$\frac{w_1}{w_2} = \frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_2} \frac{\lambda_2 - \lambda'}{\lambda_1 - \lambda'}.$$

Or if we wish to compare the resistance with that of the multiplier itself, by which it may be measured if the latter is known, we have—

$$w_1 = w_0 \frac{\lambda_0 - \lambda_1}{\lambda_1 - \lambda'}.$$

This method is only applicable to small resistances, since with large ones the damping is too feeble to be exactly measured.

The time of oscillation, and, at the same time, the damping, may be increased by the employment of an astatic pair of needles, or by weakening the earth's directive force by bringing a magnetic bar near the needle. This must, of course, have the same position in all observations. (Compare *Pogg. Ann.* Bd. 142, S. 430.)

The above formula follows at once from the equation—

$$(\lambda_0 - \lambda') : (\lambda_1 - \lambda') : (\lambda_2 - \lambda') = \frac{1}{w_0} : \frac{1}{w_0 + w_1} : \frac{1}{w_0 + w_2}.$$

Example.—It has been observed that—

with the open multiplier		$\lambda' = 0{\cdot}0025$
with the coils closed		$\lambda_0 = 0{\cdot}1435$
" "	through 1 Ohm.	$\lambda_2 = 0{\cdot}0612$
" " be measured	" the resistance w_1, to	$\lambda_1 = 0{\cdot}0978$

Then—

$$w_1 = 1 \, \frac{0{\cdot}1435 - 0{\cdot}0978}{0{\cdot}1435 - 0{\cdot}0612} \cdot \frac{0{\cdot}0612 - 0{\cdot}0025}{0{\cdot}0978 - 0{\cdot}0025} = 0{\cdot}342 \text{ Ohm.}$$

The resistance of the multiplier is—

$$w_0 = 1 \cdot \frac{0{\cdot}0612 - 0{\cdot}0025}{0{\cdot}1435 - 0{\cdot}0612} = 0{\cdot}713 \text{ Ohm.}$$

71.—Resistance of an Electrolyte.

In measuring the resistance of an electrolyte, the opposing electromotive force arising from the polarisation of the electrodes must not be neglected. The simplest method is the following modified form of that of substitution:—The fluid is supposed to be contained in a prismatic trough, of which the section is filled by the electrodes. A rheostat, a galvanoscope, and a galvanic cell, are included in the circuit.

The position of the needle is observed when so much of the fluid (and perhaps an additional quantity of rheostat resistance) is included as to reduce the deflection to a convenient amount. The electrodes are then brought nearer to each other by the length l, and rheostat resistance w is added till the needle takes the same position as before. w is then difference of resistance between the two positions of the electrodes in the column of fluid. If w be given in Siemens's units we obtain the specific resistance of the fluid compared to quicksilver as $k = \frac{wq}{l}$, where q is the section in □ mm., and l the length in metres.

Great exactness is not to be expected when the decomposition is accompanied with liberation of gas. The observation must therefore be several times repeated.

72.—Measurement of the Internal Resistance of a Battery.

I. The circuit of the battery to be examined is completed through a galvanometer, and a sufficient additional resistance is added to reduce the deflection to a convenient amount. The current-strength, which we will call J, is then observed.

Then, in the same circuit, an additional resistance, w, of known amount is included; most advantageously, such as to reduce the current-strength i, now measured, to about half J.

From these two observations, the total resistance, w, of the circuit in the first observation is obtained—

$$W = w \frac{i}{J - i}.$$

From the quantity, w, so calculated, we deduct the resistance of the galvanometer, previously measured, and also the additional resistance included in the first experiment, and so obtain that of the battery alone.

The accuracy of the result is affected by the difficulties noted in (70, I.), and is especially affected by them when the internal resistance of the battery is very small.

Example.—The resistance of a battery of 6 Daniell's elements is to be determined. The resistance of the tangent-compass and the connecting wires may be neglected. The deflection of the tangent-compass was,

including 50 units resistance, $55^{\circ}{\cdot}7$; $tan = 1{\cdot}466$;
„ 130 „ $38^{\circ}{\cdot}9$; $tan = 0{\cdot}807$;

Therefore, calling the resistance of the cell W_e—

$$W_e + 50 = (130 - 50)\ \frac{0{\cdot}807}{1466 - 0{\cdot}807} = 98{\cdot}0 \text{ units}$$

and the resistance of the cell alone $W_e = 48{\cdot}0$ units.

II. By the aid of a rheostat, and of a galvanoscope of known or negligible resistance, the internal resistance of a battery of an even number of similar cells may be obtained in the following manner :—A circuit is formed, including the galvanoscope, the battery, and a known amount of rheostat resistance, and the deflection of the needle is noted. w_1 is the total resistance of the circuit outside the battery (viz. of galvanoscope, rheostat, and connecting wires).

Secondly, the cells are arranged in pairs, with all the zincs to the same side, as shown for a battery of four cells in the annexed cut, and the needle again brought to the same deflection as before; to effect which, a different amount of rheostat resistance will be required. We take w_2 as the collective resistance of the external circuit. Then the resistance, w, of the battery in the first experiment is—

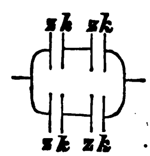


Fig. 17.

$$w = 4w_2 - 2w_1$$

For if e be the electromotive force of the battery in the first arrangement, and w its electromotive force in the second case, $\frac{1}{2}e$ will be the electromotive force, and $\frac{1}{4}w$ the resistance. Hence we have for the current-strength—

$$\frac{e}{w + w_1} = \frac{\frac{1}{2}e}{\frac{1}{4}w + w_2}; \text{ or } w = 4w_2 - 2w_1$$

This method is only applicable to very constant cells, and of considerable resistance.

III. The only possible methods of *measuring small resistances of inconstant cells* are those in which the circuit is only momentarily closed. Since in this case measurement of current-strength is impossible, we must have recourse to determination by bringing the current-strength to zero in the following way (Beetz, *Pogg. Ann.* Bd. 142, S. 573):—

Ab is a thin stretched platinum wire of known resistance, and upon which are two movable contact pieces. E is the battery of which the resistance W (in which we include the resistance of its connecting wires) is to be measured. e is another battery, of less electromotive force than E. The batteries must be connected with A by their similar poles. The contact-pieces are now moved so that no current passes through the galvanoscope G. We denote the two resistances of the platinum wire so included by a and b.

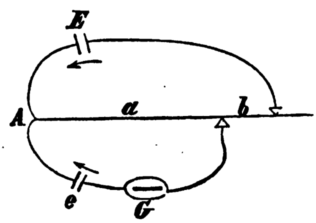


Fig. 18.

Repeating the experiment with two different resistances, a' and b', we have—

$$W = \frac{a'\,b - a\,b'}{a - a'}.$$

Proof.—Since no current passes through the branch Ge, the circuit $Aa\,bE$ must be traversed throughout by the same current. Calling this i, we have (62, II.) $E = (W + a + b)\,i$; and $e = a\,i$, and by division $\frac{E}{e} = \frac{W + b}{a} + 1$. Similarly we have $\frac{E}{e} = \frac{W + b'}{a'} + 1$; and therefore $\frac{W + b'}{a'} = \frac{W + b}{a}$, or $W = \frac{a'b - ab'}{a - a'}$.

The object of the method, the merely momentary closure of the circuit, is most easily accomplished by connecting the end of the platinum wire at A with a mercury cup. The ends of the connecting wires of E and G are amalgamated and twisted together, and are dipped for an instant into the cup and immediately withdrawn. In order that the circuit of e may not be closed alone, and so deflect the galvanoscope, the end of the wire connected with E may be allowed to project a little beyond the other.

It is obvious, from what has been said, that a must at least $= W \frac{e}{E-e}$, in order that the current in G may become 0. If it be found by experiment that no position of the contacts will answer, a feebler auxiliary battery must be substituted, or the variable resistance increased.

[*Note.*—Copper terminals are easily amalgamated by dipping them in a solution of mercuric nitrate or chloride. For Mance's method see Appendix E, 3.—*Trans.*]

73.—Comparison of Two Electromotive Forces.

In order to measure an electromotive force, we may choose that of some known constant element as a unit; that, for instance, of the Daniell cell (copper, cupric sulphate, sulphuric acid, zinc) is usually taken. In this case the measurement of an electromotive force is reduced to its comparison with that adopted as a unit. (See Table 24*c*.)

I. *Comparison with Galvanoscope and Rheostat.*—A circuit is formed, including a rheostat, a galvanoscope, and an electromotive force, which we will call E. If necessary, as much extra resistance is intercalated as will reduce the deflection to a convenient amount. The second electromotive force is then substituted for the first, and the current brought to the same amount as before by means of the rheostat. Calling

the total resistance in the first experiment W, and that in the second w, we have—

$$\frac{E}{e} = \frac{W}{w}.$$

W and w are in each case the resistances of the rheostat and that of the remainder of the circuit taken together. Especially the internal resistance of the battery itself must not be neglected. This must be measured according to the preceding article. If, however, the resistance of the rheostat be very large compared to that of the remainder of the circuit, which may always be the case by employing a very sensitive galvanometer (one of long coil), the latter may be neglected, or at least may be roughly estimated. In this case the method is both simple and convenient.

II. *Comparison by the Galvanometer.*—If two electromotive forces, E and e, produce in circuits of resistance, W_1 and w_1, the current-strengths J_1 and i_1, then—

$$\frac{E}{e} = \frac{J \,.\, W}{i \,.\, w}.$$

How we may determine, by the aid of the galvanometer, the ratio $\frac{E}{e}$ is clear without further explanation. For this the measurement of the resistance is necessary, and especially that of the battery itself.

The method becomes very simple and independent of this measurement of resistance if we make the part of the resistance constant in the two experiments very large compared to that of the batteries to be compared, so that the latter may be neglected. The current-strengths being J and i, we have simply—

$$\frac{E}{e} = \frac{J}{i}.$$

In this method we can employ only a sensitive galvanometer (tangent or sine compass with many windings, or reflecting galvanometer (65)), and always a sufficiently large included resistance.

III. *Comparison by Methods of Compensation.*—The only methods applicable to inconstant elements, of which the electromotive force varies with the current-strength, is to bring the current to zero by opposing an equal electromotive force. Poggendorff's method, which is very convenient, as it involves no measurement of internal resistance, requires the use of a galvanoscope G, a galvanometer T, and a rheostat R, and, in addition, that of an auxiliary battery S, of constant electromotive force greater than that which is to be measured. The arrangement of the experiment is shown in the figure. In the left division of the circuit is the galvanoscope G, and the electromotive force to be measured; in the right, the auxiliary battery S, and the galvanometer T. E an S are so placed that their similar poles are turned towards each other. In the middle part of the circuit, which is common to both batteries, is the rheostat R.

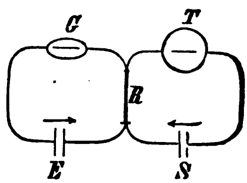


Fig. 19.

As much rheostat resistance W must now be intercalated as will cause the current in EG to vanish, and the current-strength J in T must then be observed.

The other electromotive force e must now be substituted for E, and the current in G again reduced to zero by the rheostat. The current-strength in T will now be i, while the rheostat resistance is w.

Then the proportion of the two electromotive forces is—

$$\frac{E}{e}=\frac{JW}{iw}.$$

It follows from (62, II.) that $E=JW$ when the current in the branch GE is zero.

It is sometimes a convenience to intercalate an additional resistance in the branch S. The effect of this is, that a greater rheostat resistance is required to reverse the current in G, and that the current in T is feebler.

IV. *Bosscha's Method of Compensation.*—In this arrange-

ment two rheostats (stretched platinum wires) and a galvanoscope serve to compare the electromotive force e of an inconstant element with that E of a strong and constant one. a and b are the lengths of rheostat-wires (from the connected ends to the shifting contacts) which are required in order that no current may pass through the galvanoscope G. In a second experiment this is effected by two different lengths, a' and b'. Then—

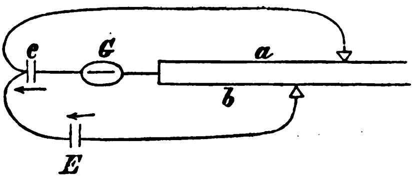


Fig. 20.

$$\frac{E}{e} = 1 + \frac{b - b'}{a - a'}.$$

The proof will be found in (72, III.)

V. The relation of electromotive forces may also be found by the experiment (72, III.), where

$$\frac{E}{e} = 1 + \frac{b - b'}{a - a'}.$$

VI. If the resistance W of the battery E and its connecting wires be known, then—

$$\frac{e}{E} = \frac{a}{W + a + b}. \quad (\textit{Dubois-Reymond.})$$

Note.—It is obvious that the Wheatstone Bridge arrangement (70, II.) may be employed both in this determination and that of (72, III.), if (by means of mercury-cups) additional resistances can be inserted at the end of the stretched wire. I will only describe in detail its application to (73, IV.) If the wires connected with the shifting contacts in Fig. 20 be connected with the ends of the stretched wire of the Wheatstone Bridge, and the galvanoscope-wire, with its movable contact-piece (as in Fig. 15), the current may be reduced to zero by shifting the contact, and a and b will then be the lengths of the wire on either side. If now an additional resistance R, of which the value in units of the bridge-wire is known, be inserted at one end, between it and the wire from one of the batteries, so as to increase its resistance, the contact will

have to be again moved to bring the current to zero, and we shall have a' the length of bridge-wire on one side, and b' that on the other, + the length of the added resistance. In 72, III. one pole of both cells must be connected with one end of the bridge-wire, and the opposite pole of E only with the other.

The value of the bridge-wire, as compared to the added resistance, is easily found by putting R and W (Fig. 15) equal, when $a = b$. If then an additional resistance R' be added at the end of b, and the current again brought to zero, we have $a' = b' + R'$, and $R' = a' - b'$. If this be repeated from each end, inequalities of the wire will be eliminated, and R' will be equal to the length of wire between the two opposite positions of the contact.

Resistance-coils are easily made of covered German-silver wire, soldered at the ends to strong copper terminals, and with the bridge may readily be adjusted to equality with a standard coil, by drawing one end of the fine wire through a drop of melted solder on the top of the terminal. A cotton reel, with terminals of strong copper wire fixed in grooves cut across each end, while the fine wire is coiled on the reel, is a convenient arrangement.—*Trans.*

74.—Determination of an Electromotive Force in Absolute Measure.

Instead of defining an electromotive force by its comparison with another of known amount, we may, by the help of Ohm's law (62, No. 4), express it in terms of current-strength and resistance. As a unit, we then employ that electromotive force which produces a unit-current in a circuit of unit-resistance. Universally, if the force E produces a current J in a circuit of resistance W—

$$E = WJ.$$

We must naturally specify in what units resistance and current-strength are measured. In the absolute system now in general use in England the unit of resistance (Weber's, *Brit. Assoc.*, or Ohm, (79), and Appendix, p. 203), is that resistance against which a unit force will perform unit work in unit time, and may be represented by a column of mercury of 1 □ mm. section, and 1034·9 mm. long. The unit of current is the magnetic unit (66).

Such a measurement may be performed by the combinations mentioned under II. and III. (previous article), if we measure not only relative but absolute current-strengths (66).

I. *Ohm's Method.*—The current-source of which the electromotive force is to be measured, is connected in circuit with a rheostat and galvanometer. We observe—

the current strength J, with included rheostat resistance W;
" " i " " w.

Then—

$$E = Ji\,\frac{w - W}{J - i}$$

To obtain exact results, it is desirable to proportion the resistances so that the current-strength in one experiment is about double that in the other.

The deviation of the deflection from the law of tangents will be eliminated if the two deflections together = 90°. If one deflection be approximately 35°, and the other 55°, this requirement will be fulfilled as well as if both had been near 45°.

The method is, of course, only applicable to constant elements; and it must be remembered that in all batteries the electromotive force is diminished by strong currents.

Example.—The electromotive force of a Grove's element is to be measured in Ohm's, or British Association units of resistance, and magnetic current measure. The reduction-factor of the tangent-compass employed is, for magnetic units, 1·586.

We observe with the resistance

$W = 10$ Ohm's, the deflection 44°·2 $\tan = 0{\cdot}9725$
$w = 20$ Ohm's, " 27°·7 $\tan = 0{\cdot}5250$

The two current-strengths are therefore in magnetic units—

$J = 1{\cdot}586 \times 0{\cdot}9725 = 1{\cdot}5424$ $\qquad$ $i = 1{\cdot}586 \times 0{\cdot}5250 = 0{\cdot}8326$

whence the required electromotive force is—

$$E = 1{\cdot}5424 \times 0{\cdot}8326\ \frac{20 - 10}{1{\cdot}5424 - 0{\cdot}8326} = 18{\cdot}09 \text{ absolute units.}$$

If the multiplication be performed with logarithms, that of the tangent is taken directly from a table. The formula $E = C. \frac{w - W}{cot\ \varphi - cot\ \Phi}$, which is identical with that employed above, is more convenient for calculation.

II. *Poggendorff's Method*, by the combination shown in Figure (p. 175).—When the current in the galvanoscope G is reduced to zero by intercalation of resistance W in the rheostat R, if J = current-strength in T, the electromotive force of the battery E is—

$$E = WJ.$$

This method is universally applicable. (Compare directions, pp. 175, 176.)

75.—Measurement of the Horizontal Force of Terrestrial Magnetism by Galvanic Methods.

As by means of a tangent-compass of known dimensions a galvanic current may be measured in absolute units if the horizontal force of terrestrial magnetism is known (66), so, conversely, the horizontal force may be determined with the tangent-compass, when its deflection is observed with a current of which the absolute strength is otherwise known.

I. *By the Voltameter.*

We pass the same current at once through a tangent-compass and a voltameter, and observe the deflection φ of the needle, and the quantity of the electrolyte decomposed per minute (regard being had to the directions in 66, 67). If r be the mean radius, and n the number of coils of the tangent galvanometer; and, further, denoting by A the number given in Table 25, last column, for the voltameter employed, the horizontal force of terrestrial magnetism is—

$$T = \frac{2n\pi A}{r} \frac{m}{tan\ \varphi},$$

On the one side, namely the current force $i = Am$ (67), and on the other $i = \frac{rT}{2n\pi}\ tan\ \varphi$ (66). By combining these two results we obtain the formula.

In case the needle and the dimensions of the section of the coils are not very small compared to the diameter of the latter, we must insert in the denominator—

$$\left(1 + \tfrac{1}{2}\frac{a^2}{r^2} - \tfrac{1}{3}\frac{b^2}{r^2} - \tfrac{3}{4}\frac{l^2}{r^2}\right)\left(1 + \tfrac{15}{4}\frac{l^2}{r^2} sin^2\varphi\right),$$

in which a, b, and l denote the same as in (66).

II. *With the Bifilar Galvanometer or Electrodynamometer.*

The bifilar galvanometer consists of a coil, suspended by two fine wires, which also serve to conduct the current. The instrument is so placed that the plane of the windings is brought into the magnetic meridian by means of the directive force of the suspending wires. When the current passes, the force of terrestrial magnetism tends to bring the plane of the coils perpendicular to the magnetic meridian with a directive force, which is proportional to the surface f, surrounded by the coils, the current-strength i, and the horizontal force of terrestrial magnetism. (Compare the Appendix, p. 200.)

The directive force D of the suspending wires, which opposes the deflection, may be found from the moment of inertia K (53) and the time of oscillation t of the coils (without current)—

$$D = \frac{\pi^2 K}{t^2}.$$

Then also if α is the angle of deflection with the current i—

$$i = \frac{D}{fT}\ tan\ \alpha = \frac{\pi^2 K}{t^2 fT}\ tan\ \alpha.$$

If now the same current produces a deflection φ in a tangent-compass of n coils of mean radius r (compare 66); then

$$i = \frac{rT}{2n\pi} \tan \varphi.$$

By combination of the two results we have—

$$T = \sqrt{\frac{\pi^2 K}{t^2 f} \cdot \frac{2n\pi}{r} \frac{\tan \alpha}{\tan \varphi}}.$$

Similarly, of course, the strength of a current, which is passed through both instruments at once, may be calculated from the deflections produced, since by elimination of T we obtain—

$$i = \sqrt{\frac{\pi^2 K}{t^2 f} \frac{r}{2n\pi} \tan \alpha \tan \varphi}.$$

By employment of a commutator, which reverses the current in both galvanometers, any inaccuracy in the position of the instrument will be compensated (63).

If the needle of the tangent-compass be suspended by a thread of ratio of torsion Θ (64), we must write $r\ (1 + \Theta)$ instead of r throughout.

The current in one galvanometer ought to exert no deflecting force on the other. (Compare *Pogg. Ann.*, Bd. 138, S. 1.)

76.—Methods of Measuring Currents of Short Duration by Multiplication and Recoil.

In measuring currents of short duration with a damped needle (50), especially, for instance, in the measurement of induced currents, it is often advantageous to repeat the impulse at regular intervals. Hence from the damping of the needle there finally results a constantly maintained movement (exactly like that of a clock pendulum, which at each swing receives an impulse from the driving weight), but by friction, and the resistance of the air, is so damped that a series of swings maintains a constant amplitude.

Therefore, if this final result be employed, we obtain an observation easily repeated, and from which an exact mean

can be taken; and, further, it is not important that the needle should be at rest at the commencement of the experiment.

I. *Method of Multiplication.*

The proceeding is quite analogous to the example of the clock pendulum already adduced. An impulse is imparted to the needle, which swings out, and turns back. At the instant when it has passed its position of equilibrium backwards, a second impulse is imparted in the opposite direction to the first, so that it increases the motion of the needle. At the following passage through the point of equilibrium, another impulse is imparted in the same direction as the first, and so on. The oscillation is each time wider, till it reaches an amplitude at which that given by previous impulses is only just maintained, and of course this limit is the sooner reached the stronger the damping.

Assuming that small oscillations are employed, which are observed by the mirror and scale, the limiting arc is proportional to the increase of velocity through a single impulse, which is also proportional to the quantity of electricity passing through the multiplier in a current of short duration. Mostly this proportionality is sufficient, as for instance in the following article.

We may also calculate the arc of oscillation α, which the needle previously at rest receives from a single impulse without any damping, from the limiting angle obtained by multiplication, as soon as the ratio of damping $\lambda = log\ k$ is known (50). The theory of the swinging damped needle shows, namely, that—

$$\alpha = A\left(1 - \frac{1}{k}\right) k^{\frac{1}{\pi}\ arc\ tan\ \frac{\pi}{2 \cdot 3026\lambda}}$$

($2{\cdot}3026\lambda$ is = nat. log. k). For small values the latter limb of the equation approximates to the value $\sqrt{k}$, and we have—

$$\alpha = A\,\frac{k-1}{\sqrt{k}}.$$

The angular velocity v, communicated to the needle by a single impulse, t being the time of oscillation of the needle, is—

$$v = \frac{\alpha}{2} \frac{\pi}{t}.$$

II. *Method of Recoil.*

This method, which is employed with powerful impulses, yields at the same time the ratio of damping of the needle.

The needle is set in motion by a single impulse, and is allowed to swing out, back, and out in the opposite direction; then, at the instant of again passing its position of equilibrium (scale-division, which the needle indicates when at rest), a second impulse is given it in the opposite direction to the first. By this the needle will be thrown back again, since it has lost velocity by the damping. It is now allowed again to return twice, and again thrown back at the moment when it next passes its position of equilibrium, and so on. When this proceeding has been several times repeated, the throw of the needle takes a constant value, and indeed does so the sooner, the stronger the damping. When this is the case, the oscillations are of the form graphically represented in the diagram annexed, in which the times are the abscissæ, and the scale-divisions, reckoned from the position of equilibrium of the needle, are the ordinates.

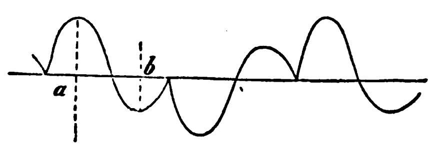


Fig. 21.

The establishment of these regular oscillations will be hastened if the first impulse be enfeebled, and the more so the feebler the damping is. If indeed there were no damping, the first impulse should only amount to half the succeeding ones, as follows from the figure.

The method of recoil yields, on taking the mean of the corresponding observations, 4 turning points on the scale. The difference a between the two outer we will call the

great arc of oscillation, the difference b of the two inner the smaller arc. (See Fig. 21.)

Then directly the ratio of damping is—

$$k = \frac{a}{b}.$$

The angular velocity communicated by a single impulse is proportional to—

$$\frac{a^2 + b^2}{\sqrt{ab}},$$

when the damping is small, or only varies slightly in amount; as for instance in comparison of resistances (78). Otherwise the factor $k - \frac{1}{\pi}$ *arc tan* $\frac{2{\cdot}3026\lambda}{\pi}$ must be employed.

To obtain the angular velocity v itself, we must, in addition, know the period of oscillation t of the needle. Then—

$$v = \frac{1}{2} \frac{\pi}{t} \frac{a^2 + b^2}{\sqrt{ab}} k^{-\frac{1}{\pi} \, arc \, tan \, \frac{2{\cdot}3026\lambda}{\pi}}.$$

Compare, respecting the methods of multiplication and reversal, W. Weber, Electrodynamische Maassbestimmungen insbesondere Widerstandsmessungen—*Abh. d. K. Säch. Ges. d. Wiss.*, 1, S. 341. Also J. C. Maxwell, *Treatise on Electricity*, vol. ii. para. 750, 751.

If the angles of oscillation are so great that the proportionality between angles and scale-readings no longer holds, the latter must be reduced (48) to the sine of the half-angle of deflection from the point of equilibrium, since this, as in the pendulum, is proportional to the velocity with which it passes its position of equilibrium. If an observed arc of oscillation $= n$ scale-divisions, we have the magnitude $\frac{11}{128} \frac{n^3}{r^2}$; r being the distance of the mirror from the scale.

77.—Measurement of the Inclination of Terrestrial Magnetism by the Earth-Inductor—(*Weber*).

This measurement rests on a comparison of the currents produced by the horizontal and vertical components in the same revolving coil (inductor).

Since the scale-readings of the galvanometer (when large reduced to the sine of $\frac{1}{2}$ deflection to one side; compare end of previous article) are proportional to the current-strengths, and the latter to the required components, the ratio of the scale-readings gives the tangent of the angle of inclination.

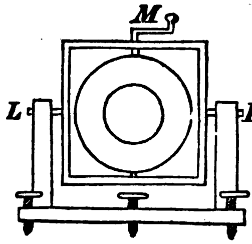


Fig. 22.

The earth-inductor consists of a rotatable coil, of which the axis of rotation M may be placed horizontally or vertically. An "induction-impulse" will be caused by rapidly turning the coil through 180°, the plane of the coil both before and after the rotation being perpendicular to the required component of terrestrial magnetism.

To measure the current produced by this rotation, a galvanometer, with a suspended needle, which has a time of oscillation of at least 10 seconds, should be employed. Ordinarily a double astatic needle is used. The narrow coils of the multiplier serve to damp the needle, and if these are not sufficient, it is strengthened by a copper casing inserted in the multiplier.

In observation the multiplication method (76) will usually be employed, as we assume in the following. It is only with very powerful inductors that the use of the method of reversals is practicable.

Induction by the Vertical Component.—By turning LL, the coils are placed horizontally, and, by the aid of a magnetic needle, the axis M is brought into the magnetic meridian.

The axis LL must next be carefully levelled by means of the foot-screws, and a spirit-level placed upon it. The position of this axis must not be afterwards changed, and any further correction must be made only with the screw at the back (shown in the middle of Fig. 22).

We must now set the axis of rotation M of the coils exactly horizontal—that is, so that the bubble in a spirit-level placed upon it keeps the same position in the tube when it is

turned end for end. Now a set of induction observations must be made, according to I., previous article, in which, for each impulse, the coil must be turned 180°. The arc of oscillation finally produced we will denote by A_1.

Induction by the Horizontal Component.—The inductor is placed as shown in Fig. 22—that is, the coils are placed vertically, and turned to one of the stops, and a spirit-level is placed on the top of the axis M, so that its tube is in the magnetic meridian. The central foot-screw is now turned till the position of the air-bubble is unchanged in the tube by turning the coils 180°. When this is the case, the axis M is in a vertical plane perpendicular to the magnetic meridian.

We now make a second set of observations precisely as before, and call the constant arc of oscillation A_2.

Then the J is given by the formula—

$$\tan J = \frac{A_1}{A_2}.$$

Testing of the Instrument.—It is most easily known in the second position of the coils when the two opposite-sided positions given by the two stops really differ by 180°, if it be provided with a small plane mirror, silvered on both sides. The axis M is placed perpendicularly, and the mirror also placed perpendicularly upon it, and the eye brought to the same height as the mirror, and perhaps a metre distant, so that a vertical mark (*e.g.* a window-bar) is visible in it. On rotating to the second position the mark must again appear.

A second test consists in proving that the plane of the coils, when resting against the stops, is perpendicular to the magnetic component to be measured. In a geometrical inductor wound carefully in a frame this may be determined for the horizontal component by a compass with right-angled case held against the frame, and for the perpendicular by the spirit-level. In other cases the annexed (Fig. 23) arrangement may be employed, which is fixed to the stops, and limits the rotatory play to perhaps 30°. With this limited angle of rotation, a set of induction obser-

Fig. 23.

vations is made on each side, of which the resulting final deflections should be similar.

A slight error (perhaps 1°) in the fulfilment of both conditions only causes a vanishing error in the result, and to provide against it the greatest care must be taken in adjusting *MM* with the spirit-level.

Compare W. Weber on the Employment of Magnetic induction for Measurement of Inclination—*Abh. d. Gött. Ges. d. Wiss.*, Bd. 5, 1853.

78.—Comparison of Two Resistances with the Magneto-Inductor.

The magneto-inductor of Weber consists of a solenoid, through which a bar, consisting of two magnets, with their like poles opposed, can be pushed. Pushing this through from one stop to the other produces an electromotive force in the coils differing in direction with that of the motion of the bar, but always similar in amount. The end positions are so regulated by movable stops, that near them a slight motion produces no electromotive force. At each side there is such a position, which may easily be found by experiment.

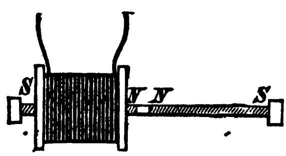


Fig. 24.

If the ends of the inductor-coil are connected by a conductor, a certain quantity of electricity will pass through it at each inductive impulse, and this quantity with the same inductor is solely dependent on the collective resistance of the circuit (solenoid and external conductor) to which it is inversely proportional. If a galvanometer, with suspended astatic needle of suitable period of oscillation, form part of the circuit, this quantity may be measured either by the multiplication or recoil method (76). It is best to employ the latter, which is independent of the variation of damping produced by the intercalated resistance.

In order to compare two resistances, w_1 and w_2, it is necessary to make three sets of observations, namely—

(1.) In which the inductor-coil and galvanometer alone form the circuit. The great and little arcs are a and b.

(2.) In which the resistance w_1 is included. The arcs are a_1 and b_1.

(3.) In which w_2 is substituted for w_1. The arcs are a_2 and b_2.

Denoting, then, the formula $\frac{a^2+b^2}{\sqrt{ab}}$, etc., by i, i_1, and i_2, we have (compare p. 166)—

$$\frac{w_1}{w_2} = \frac{i - i_1}{i_1 - i_2} \frac{i_2}{i_1}.$$

79.—Absolute Measurement of Resistance.

We employ an earth-inductor (77), of which the area of surface surrounded by the coils is known, and a galvanoscope, of which the wire is wound on a wooden frame, and closely surrounds, and therefore damps, the needle. The time of oscillation of the latter must be at least 15 seconds, and its sensitiveness sufficient to permit of the employment of the method of recoil. Taking

K the moment of inertia of the galvanometer needle (53);
t its time of oscillation;
S the sum of the surfaces surrounded by the coils of the inductor in □ mm.;
T the inducing component of the earth's magnetism;
a and b the two arcs of oscillation (p. 183) taking as unit an arc equal in length to its radius;
λ = nat. log. a – nat. log. b, the natural logarithmic decrement of the needle with closed circuit;
λ' that with interrupted circuit (compare 70, III.);

then the absolute resistance w of the closed circuit is—

$$w \; \frac{32S^2T^2}{K} \frac{t}{\pi} \frac{\lambda - \lambda'}{\sqrt{\pi^2 + \lambda^2}} \frac{ab}{(a^2+b^2)^2} \left(\frac{a}{b}\right)^{\frac{2}{\pi} \tan^{-1} \frac{\lambda}{\pi}}.$$

Compare p. 201, *et seq.*; and W. Weber upon Galvanometers—*Abh. d. Götting. Ges. d. Wiss.*, Bd. 10; also *Reports of Com. of Brit. Assoc.* 1862, *et. seq.*

APPENDICES.

APPENDIX A.

The Absolute System of Electrical Measurement.

Every kind of magnitude requires for its measurement —that is, its numerical expression—some unit of the same nature as itself. This unit is at first arbitrary, and may be defined, for many kinds of magnitude, as a preserved original measure (scale, standard); but in other cases—as, for instance, velocity, or quantity of heat, or electricity—such a definition is impossible. Hence, such magnitudes are expressed by means of geometrical and physical laws, in terms of other quantities which can be so defined; as when, for instance, we select as units the *velocity* with which a body traverses unit length in unit time; that quantity of *heat* which will raise one unit of mass of water 1° in temperature; and that quantity of *electricity* which will exert unit force upon a similar quantity at unit distance. As distinguished from the arbitrary or primary measures, we may call the latter "derived measures."

The introduction of such measures, unavoidable in the first instance, will be seen on further consideration to be also very advantageous. For it is obvious that the diminution of the number of arbitrary primary measures in itself indicates an advance, while the new units may be so chosen that the natural laws which they serve to define are expressed by them in a simpler form. For instance, the space l traversed by a moving body is universally proportional to the velocity u and the time t, or $l = \text{Const. } ut$—the numerical value of the constant depending on the units selected. Should we take as unit of velocity that of a falling body at the end of the first second, this constant $= g$. According to the defini-

tion previously given, however, the constant = 1, and the law will take its simplest form, $l = ut$.

The geometrical relations are similarly simplified if we employ for the measurement of area and contents, instead of arbitrary units, the square and cube of the unit of length, an advantage of which science has always availed itself, but which is not yet fully carried out in common life.

In this manner each "derived unit" serves to eliminate the constant of a natural law.

Among the objects to which preserved elementary standards are inapplicable we may count almost all magnetic and electrical quantities, and hence we have here a specially prominent application of the system of derived units. This application was carried out by Gauss and Weber, who showed that all these quantities might be expressed in units of length, mass, and time. Units deduced in this manner are specially called absolute measure.*

The choice of primary units of length, mass, and time, is in the first instance entirely arbitrary; but, following the example of Gauss, the *millimetre* is taken as *unit of length*, the *second* as *unit of time*, and the *milligramme* as *unit of mass*, except where otherwise stated.

It must also be distinctly understood that, in this case, a milligramme means the *mass* of 1 cubic millimetre of water; whilst, in popular language, grammes, etc., are usually spoken of as *weights*. For example, the moment of inertia of a small body of m $^{\text{mgr.}}$, and distant a $^{\text{mm.}}$ from an axis of rota-

* The term "absolute" was first applied in this manner to the unit of intensity of terrestrial magnetism defined by Gauss. In opposition to the arbitrary practice, previously common, of taking the intensity at London as unity, and making other observations merely relative to this, Gauss gave in his *Intensitas vis magneticæ terrestris ad mensuram absolutam revocata* an absolute (that is, not a merely comparative) unit for terrestrial magnetic intensity, deduced from the primary units of length, mass, and time only, and applicable to magnetic quantities in general. In a similar manner W. Weber, in supplying the need of independent, and not merely comparative measures, for the various electrical quantities, has retained the same designation.

The name "absolute measure" has now become a scientific phrase of determinate meaning, and must therefore be unconditionally retained, although it must be admitted that the term "derived measure" hits more exactly the essential point of the system.

tion, is, in the absolute electromagnetic system, $= a^2m$, and not $a^2 \frac{m}{g}$. On this account the moment of rotation, which it experiences from the attraction of the earth, at a horizontal distance a from an axis of rotation $= amg$, where by g we denote the acceleration by gravity, measured in mm. and sec., which at latitude $45° = 9806$.*

According to what has just been said, all magnitudes are represented as functions of length, time, and mass; a velocity, for instance, as a length divided by a time, a volume as third power of a length, a force as a length multiplied by a mass and divided by the square of a time. In the following article we will define this function with regard to each magnitude, and, at the same time (after the example of Maxwell and Jenkin—*Rep. Brit. Assoc.*, 63, p. 132), will take the dimensions of the related magnitude. Throughout we denote a length by l, a time by t, and a mass by m. The dimensions of a space are $= l^3$, of a velocity $= lt^{-1}$, of a force mlt^{-2}.

These dimensions give the power to change from the constant arbitrary units, mm., mgrm., and sec. (when they are inconveniently small, or introduce an unmanageable number of figures), to others more convenient. If, then, a primary unit occurs in the deduced unit in the nth power, the deduced unit is changed in the ratio k^n if the primary

* If a general answer be desired to the question whether grammes, etc., have to serve as units of mass or weight, there can be little doubt as to the scientific reply :—That as the weight of a body is clearly entirely indeterminate, and is even variable on the earth's surface to the extent of $\frac{1}{2}$ per cent, the weight of a body can never serve as a unit of weight. It would also be wrong to say that as unit of weight we take a cubic centimetre of water at 45° latitude; since then a set of weights would have to be specially adjusted for each degree of latitude. What is really meant by the phrase "set of weights" is nothing but a set of masses; and a weighing with an ordinary balance is no measurement of weight, but one of mass. The weight, that is the force with which a body is attracted by the earth, is obtained by the measurement of velocity of falling; as, for instance, by the time of oscillation of a body suspended by a thread.

In fact also the aim of weighing is generally measurement of mass. The chemist, the merchant, and the doctor, have nothing to do with the pressure of a body on what supports it, but solely with its mass, for to this its chemical power, its nutritive or its money value, is proportional.

be changed in that of k. The numerical value of the magnitude thus expressed will be changed in the ratio k^{-n}. The number representing a velocity, will, by the change from mm. to cm. as units of length, be changed itself in the ratio 10^{-1}, by change from sec. to min. in that of 60^{+1}. The numerical value of a force, expressed in cm. and grm., instead of mm. mgrm., will be diminished in the ratio $10^{-1} . 1000^{-1} = \frac{1}{10000}$.

Turning now to individual measures and measurements, we will first examine the elementary mechanical units on which absolute magnetic and electrical measurements depend.

Mechanical Measures.

Force.—By an elementary law of mechanics, the velocity u communicated by the force k to a body of mass m in time t, is given by the formula $u = C \frac{kt}{m}$, the constant c being dependent on the unit selected. Taking $C = 1$, and so giving the law its simplest form, and letting v, t, m, and k also $= 1$, we have—

> Unit of force; that force which in unit time communicates unit velocity to a unit mass. Dimensions $= lmt^{-2}$.

The force exerted on m mgr. by the earth's attraction $= 9806\ m \frac{\text{Mm. Mgr.}}{\text{Sec.}^2}$.

Work.—Work will be performed when the point of application of a force is moved by it. The performed work A is proportional to the force k, and to the distance l, over which movement has been performed. If we take the law in its simplest form, and set work as the product of force and distance, $A = kl$, then

> the unit of work is performed when a point, acted on by unit force, is moved by it through unit distance. Dimensions l^2mt^{-2}.

In raising 1 grm. 1 metre, the work $1000 \times 9806 \times 1000 = 9806 \times 10^6 \frac{\text{Mm.}^2\ \text{Mgr.}}{\text{Sec.}^2}$ is performed.

Moment of Rotation.—Taking the moment of rotation, P, as the product of a force k into the length of its leverage l (that is, its distance from axis of rotation), $P = kl$; then

> The unit of moment of rotation is given by unit-force acting through a lever of unit-length. Dimensions, l^2mt^{-2}.

Directive Force.—If a body, movable round a fixed axis, has a stable position of equilibrium, a moment of rotation is exerted on it in any other position, which, for a small angle of deflection φ, is always proportional to φ. We therefore obtain the constant ratio $\frac{P}{\varphi} = D$ as the directive force, our unit of angular measure being that angle subtending an arc equal to the radius ($= 57°{\cdot}296 =$ arcual unit).

The unit of directive force is obtained when the moment of rotation for a small deflection from the position of equilibrium is equal to the angle. Dimensions $= l^2mt^{-2}$.

The directive force of a pendulum moved by gravity, of which the mass is m mgr., and the length l mm. from the point of suspension to the centre of gravity, is therefore $lm\ 9806 \frac{\text{mm.}^2\ \text{mgr.}^2}{\text{sec.}^2}$, for the moment of rotation for a deflection $\varphi = lm\ 9806\ \sin\varphi$, and for small angles, φ may be taken as $= \sin\varphi$.

Moment of Inertia.—Taking the moment of inertia K of a mass m at the distance l from its axis of rotation, $K = l^2m$, or if more masses be present, $K = \Sigma l^2m$; then

> The unit of moment of inertia is represented by a point, of unit-mass, at unit-distance from an axis of rotation. Dimensions $= l^2m$.

The moment of inertia of a magnet hung from a thread, and of length l and width l' mm., and mass m mgr., is therefore (53)

$$K = m \frac{l^2 + l'^2}{12} \text{ mm.}^2 \text{ mgr.}$$

Moment of inertia K, directive-force D, and time of oscillation t for small arcs, are connected by the equation $\frac{t^2}{\pi^2} = \frac{K}{D}$, as the dimensions themselves show, since l^2m divided by l^2mt^{-2} gives the square of a time.

3. Electrostatic Measure.

Electrical Quantity.—Two quantities of electricity, ϵ, ϵ', considered as concentrated in points, and at the distance l, repel each other with the force $k = \text{Const.} \frac{\epsilon\epsilon'}{l^2}$, in which the numerical value of the constant depends on the unit selected. Putting the constant = 1, and so giving the law its simplest form, $k = \frac{\epsilon\epsilon'}{l^2}$; the so-called mechanical

> Unit of electrical quantity is that quantity which repels an equal quantity at unit-distance with unit-force. Dimensions = $l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-1}$.

4. Magnetic Measure.

Quantity of Free Magnetism, or Strength of Magnetic Pole.—Exactly as above, we may write the elementary law of the interaction of two hypothetical quantities μ μ' of free magnetism (or two magnetic poles of the nature of points, of strengths μ and μ') at the distance l, $k = \frac{\mu\mu'}{l^2}$ and so obtain as

> Unit-quantity of free magnetism (or strength of unit-pole), that quantity or pole which exerts unit-force on a similar one at unit-distance. Dimensions, $l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-1}$

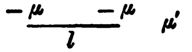
Magnetism of Bar, or Magnetic Moment.—Each magnet has equal quantities of free positive and negative magnetism.

The simplest bar-magnet would consist of two opposite poles of the nature of points, and of equal strength. If $\pm\mu$ be the quantity of magnetism which is contained in each pole, and l the distance between them, the action of the bar at a distance will be proportional to $l\mu$. Taking $l\mu$ as the magnetic moment, or, shortly, as the magnetism of the bar,

> A magnet which consists of two poles, with the quantity ± 1 of free magnetism (or of unit-strength), and separated by unit-distance, represents the unit of strength of a bar-magnet. Dimensions, $l^{\frac{5}{2}}m^{\frac{1}{2}}t^{-1}$.

According to theory, the magnetic moment imparted by the same magnetising force to two bars of the same form is proportional to their masses. The maximum of permanent magnetism which very thin rods can retain is about 1000 $\dfrac{\text{mm.}^{\frac{5}{2}}\ \text{mgr.}^{\frac{1}{2}}}{\text{sec.}}$ for each milligramme of steel. The relation of the magnetic moment to the mass in milligrammes is called the specific magnetism of a bar. The force exerted by a magnet on a magnetic pole is given by the following considerations:—

(1.) The magnetic pole μ' lies in a line passing through the two poles (first position of Gauss), its distance from the centre of the magnet being L. The nearer pole exerts a force



$\dfrac{\mu\mu'}{\left(L-\frac{l}{2}\right)^2}$, and the further pole a similar force in the opposite direction $=\dfrac{\mu\mu'}{\left(L+\frac{l}{2}\right)^2}$; and the total resultant force, attractive or repulsive, according to whether the opposite or similar pole is the nearer, amounts to—

$$k=\mu\mu'\left(\frac{1}{\left(L-\frac{l}{2}\right)^2}-\frac{1}{\left(L+\frac{l}{2}\right)^2}\right)=\mu\mu'\,\frac{2Ll}{\left(L^2-\frac{l^2}{4}\right)^2};$$

but $l\mu$ is the magnetism of the bar $= M$, and therefore

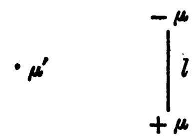
$$k = 2M\mu' \frac{L}{\left(L^2 - \frac{l^2}{4}\right)^2} = 2\,\frac{M\mu'}{L^3}\left(1 - \frac{l^2}{4L^2}\right)^{-2} =$$

$$k = 2\,\frac{M\mu'}{L^3}\left(1 + \tfrac{1}{2}\,\frac{l^2}{L^2} \ldots\ldots\right).$$

If the distance L be very great compared to l, so that $\frac{1}{2}\,\frac{l^2}{L^2}$ may be neglected in comparison with 1, we have simply

$$k = 2\frac{M\mu'}{L^3}.$$

(2.) The magnetic pole μ' is placed on a line perpendicular to the axis of the magnet, and passing through its centre (second position), and at the distance L from the middle of the magnet. The dissimilar pole exerts an attractive force $= \frac{\mu\mu}{L^2 + \frac{l^2}{4}}$, and the similar pole a repulsion of like amount.



Both forces are resolved, according to the parallelogram of forces, into a single force acting parallel to the axis of the bar, viz.—

$$k = 2\,\frac{\mu\mu'}{L^2 + \frac{l^2}{4}} \cdot \frac{\frac{1}{2}l}{\sqrt{L^2 + \frac{l^2}{4}}} = \frac{l\mu\mu'}{\left(L^2 + \frac{l^2}{4}\right)^{\frac{3}{2}}} = \frac{M\mu'}{\left(L^2 + \frac{l^2}{4}\right)^{\frac{3}{2}}}.$$

$$k = \frac{M\mu'}{L^3}\left(1 + \frac{l^2}{4L^2}\right)^{-\frac{3}{2}} = \frac{M\mu'}{L^3}\left(1 - \frac{3}{8}\,\frac{l^2}{L^2} + \ldots\ldots\right).$$

For a great distance L, $k = \frac{M\mu'}{L^3}$.

If we replace the magnetic pole μ' by a short magnetic needle, at right angles to the direction of the force, and of the length l', and of which each of the poles has the strength μ', a couple will be produced exerting a moment of rotation $2k\frac{l}{2} = kl'$ upon it. Since $\mu' l'$ is the magnetic moment of the needle M', the moment of rotation exerted on it by another magnet M at the distance L (great compared with the length of the magnets) will be—

In the *first position* (when M' lies in the produced axis of M, and at right angles to it, $P = 2\,\frac{M'M}{L^3}$.

In the *second position*, that is, when M' lies on the perpendicular to M, and is itself at right angles with M, $P = \frac{MM'}{L^3}$.

Hence we may express the unit of bar magnetism independently of the definition of single poles, but entirely corresponding with the above, in the following manner:—

> The unit of bar-magnetism is possessed by a bar which exerts on a similar bar at the (great) distance L, in the *second position* (compare above), the moment of rotation $\frac{1}{L^3}$.

If the deflected magnet makes an angle φ with the direction of the force, the moment of rotation will obviously be obtained by multiplying the above result by *cos* φ.

What is here indicated for ideal magnets, with points for poles, is also true of the actual. For, in action at a distance, there are two mean points in which we may consider the positive and negative magnetism to be concentrated. Ordinarily, the positions of these "poles" are not exactly known, on account of which (58, II.), in case a distance is employed at which the correction with $\frac{l^2}{L^2}$ is not inappreciable, we must repeat the observation at a second distance in order to eliminate it.

Intensity of Terrestrial Magnetic Force.—At any point of the earth's surface a magnetic pole is acted on by a force proportional to the strength μ of the pole. We take the force which is exerted on a unit-pole as the intensity of terrestrial magnetic force at the place, or, shortly, as the intensity of terrestrial magnetism. Horizontal intensity T is the horizontal component of this force, which alone acts on an ordinary needle. For the sake of brevity we will confine our observations to this portion.

Since the force acting on a pole μ is given by μT, the moment of rotation exerted on a magnetic needle at right angles to the direction of the force, and with two poles $\pm\mu$ distant l from each other, will be $2\mu T\ \frac{l}{2} = \mu l T = MT$, if M denotes the magnetic moment of the needle. We have therefore

> As unit of terrestrial magnetic force, that intensity which exerts a unit moment of rotation on a bar of unit magnetic moment at right angles to the direction of the force. Dimensions, $l^{-\frac{1}{2}}m^{\frac{1}{2}}t^{-1}$.

Supposing that the magnet make with the direction of the force the angle φ, we have a moment of rotation of $MT\ sin\ \varphi$. So also MT is for a movable magnet that magnitude which we have previously called directive force, and it determines, therefore, the equation $\frac{t^2}{\pi^2} = \frac{K}{MT}$ for time of oscillation t, and moment of inertia K, from which we obtain the product MT of magnetic moment and horizontal intensity (58, I.)

The angle through which a short magnetic needle is deflected from the magnetic meridian by another magnet is obtained as follows:—

The magnet M is placed in the "first position" (58, II.) to the needle, of which the moment is M', and distance L. If φ be the angle of deflection, the moment of rotation exerted by the magnet for this angle $= 2\ \frac{MM'}{L^3}\left(1 + \frac{1}{2}\ \frac{l^2}{L^2}\right)$ $cos\ \varphi$, which is equal to the $M'T\ sin\ \varphi$ exerted by the terrestrial magnetic force. Therefore—

$$tan\ .\ \varphi = \frac{2}{L^3}\ \frac{M}{T}\left(1 + \frac{1}{2}\ \frac{l^2}{L^2}\right),$$

an equation which is employed in the proof to the determination of $\frac{M}{T}$. The magnitude there denoted by a has the physical signification that $\sqrt{2a}$ is the distance between the

two poles of the magnet. In the "second position" the factor 2 disappears, and instead of $\frac{1}{2}l^2$, we have $-\frac{3}{8}l^2$.

5. Galvanic Measure.

Current-Strength—Mechanical Measure.—The numerical value for a current-strength is given in the most direct manner by the mechanical measure of the quantity of electricity (p. 194) which passes through a section of the circuit in unit of time, and hence the mechanical

> Unit of current-strength is that current by which, in unit time, unit quantity of electricity passes through a section of the circuit. Dimensions $= l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-2}$.

This unit of current is practically unemployed on account of the great difficulty of such a measurement; but we make use of an effect of the current to define the current-strength, employing mostly its chemical or magnetic action.

Chemical Current-Measure.—Here the

> Unit is that current which in unit time effects unit chemical action.

In reality this measure is not absolute in the full sense, for the quantity of an electrolyte decomposed by the current is dependent on the nature of the substance; and hence, not only on units of mass, length, and time, but on an arbitrary quantity of the substance employed. Since the decomposition of equivalent weights is proportional, and since the chemist takes that of hydrogen as unit, we employ in current-measurement the separation of a unit of hydrogen as unit of chemical action. It is customary in practice to reckon the amount of decomposition either in mgr. of water or in c.c. of mixed gases, measured at 0° and 760° mm. (compare 67).

Magnetic (or Weber's) Current-Measure.—If we consider the action of a rectilinear portion of a current of length l and of strength i on a quantity μ of free magnetism, at a

distance L from the current-element, measured at right angles to its direction, we find the transverse force exerted by the magnetic pole on the current, or, conversely, by the current on the pole, to be $k = \text{Const.} \frac{li\mu}{L^2}$. Expressing the law in its simplest form, we have $k = \frac{li\mu}{L^2}$. and as

> Unit of current-strength, that current which, under the above normal relations, exerts unit-force on a unit magnetic pole. Dimensions, $l^{\frac{1}{2}}m^{\frac{1}{2}}t^{-1}$.

Instead of this we may say, without assertion of the real existence of the current—

> The unit-current in a circle of the radius L, surrounding a short needle of unit-magnetism, which lies in the plane of the circle, exerts on the needle a moment of rotation $\frac{2\pi L}{L^2} = \frac{2\pi}{L}$.

According to Ampère's law of the reciprocal action of two currents, the following definition is identical with the above:—

> Two rectilinear parallel unit-currents of unit-length flowing in the same direction attract each other at the (great) distance L with a force $\frac{2}{L^2}$.

Lastly, we have the relation for a plane surrounded by a current, that, in regard to the distant magnetic effect exerted or experienced by it, it behaves like a magnet passing through its centre and perpendicular to its plane, and of the magnetic moment $f\,i$, where f is the magnitude of the conducting plane. As unit of surface, the square of the unit of length is of course employed. In other words, we may say—

> A unit-current passing round a unit-surface behaves to other currents or magnets like a short magnetic bar of unit magnetic moment, and perpendicular to the plane of the surface.

The above rule may easily be applied, for instance, to a circular current which acts on a magnetic pole μ lying in its axis. Let the current-strength be i, the radius of the circle l, and the distance of the poles from the plane of the circle L. Each small portion of the circle of length λ exerts the force $\frac{\lambda i\mu}{L^2+l^2}$. All these forces are resolved into a force acting from the centre of the circle, their components in the plane of the circle being eliminated. We need therefore only sum the components perpendicular to this plane to obtain the total force. The component resulting from λ is—

$$\frac{\lambda i\mu}{L^2+l^2}\cdot\frac{l}{\sqrt{L^2+l^2}}=\frac{\lambda l i\mu}{(L^2+l^2)^{\frac{3}{2}}}.$$

Since the whole circumference $=2\pi l$, the total force will be $\frac{2\pi l^2 i\mu}{(L^2+l^2)^{\frac{3}{2}}}$. πl^2 is the plane surrounded by the current. For a great distance l^2 may be neglected in comparison to L^2, and the force becomes $\frac{2fi\mu}{L^3}$; that is, the current acts exactly as a magnet of magnetic moment fi. We may take therefore fi as the magnetic moment of the current surrounding the plane f. (Compare 75, II.)

As to the relation between chemical and magnetic measure we have spoken in (67).

Electromotive Force.—The absolute measure for this magnitude is deduced from the phenomena of magneto-induction. The law may be stated in its simplest case as follows:—In a field of uniform magnetic intensity T, we have a rectilinear conductor of length l, and perpendicular to the direction of T. This is moved at right angles to the plane of l and T with a velocity u. By this motion an electromotive force e is induced in the conductor proportional to the length l, the magnetic intensity T, and the velocity u. Taking simply $e=lTu$, we have as

Unit that electromotive force which is induced in a rectilinear conductor of unit length, moving with

unit velocity across a unit magnetic field in a direction at right angles to itself, and to the magnetic force. Dimensions $l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-2}$.

If, for instance, in Central Germany, where the total magnetic intensity = 4·5, we hold a straight wire of 1000 mm. length perpendicular to the line of dip, and move it perpendicular to itself, and to the magnetic dip, with a velocity of 1000 mm. per second, the induced electromotive force $= 1000 \times 4{\cdot}5 \times 1000 = 4500000 \dfrac{\text{mm.}^{\frac{3}{2}}\ \text{mgr.}^{\frac{1}{2}}}{\text{sec.}^2}$.

In this absolute measure the electromotive force of a Daniell's cell is 114×10^9, and of a Grove's 194×10^9 $\dfrac{\text{mm.}^{\frac{3}{2}}\ \text{mgr.}^{\frac{1}{2}}}{\text{sec.}^2}$. This measure is most advantageously realised in practice by the electromotive force induced in a revolving coil by the earth's magnetism.

This will be given in correspondence with the above definition by the following rule:—We figure the coils as projected on a plane perpendicular to the direction of the earth's magnetism. The sum of the surfaces, surrounded by all the coils, changes its amount during the revolution at a certain instant by the small unit df in the short time dt. At this instant, therefore, the induced electromotive force in absolute measure = the magnetic intensity multiplied by the velocity $\dfrac{df}{dt}$ of the change of plane; and $e = T\,\dfrac{df}{dt}$.

The wire passes twice over the area of coil in each rotation, and therefore $e = \dfrac{2r^2\pi T}{t}$, where r = radius of coil, and t time of rotation.—*Trans.* (Compare *Brit. Assoc. Rep.*, 1863.)

Finally, the same unit of electromotive force is obtained from the law of magnetic induction, deduced in the following universal form from the electro-magnetic force. Suppose a wire of any form moving in the neighbourhood of a magnet with the velocity u. In order to obtain the induced electromotive force, we suppose the conductor to be traversed by a unit-current of Weber's measure (p. 199). A motive force will then be exerted upon the conductor, and k being its

component at any instant in the direction of the actual motion, the induced electromotive force at that instant is $e = -ku$. In the case of circular motion k is the component of the moment of rotation in the plane of revolution, and u the angular velocity.

We have employed another definition of electromotive force (74) dependent on the units of current and resistance—namely, writing Ohm's law $i = \frac{e}{w}$, we have as unit that electromotive force which produces unit-current in a circuit of unit resistance.

Resistance to Conduction.—In the absolute system of measurement we make use of Ohm's law to obtain a unit of resistance from the units of current and electromotive force, and take

> As unit the resistance of a conductor, in which unit electromotive force produces a unit-current. Dimensions $= lt^{-1}$.

In this measure the resistance of a column of mercury 1 m. long, and 1 □ mm. section (Siemens's unit) $= 9705 \,.\, 10^6 \frac{\text{mm.}}{\text{sec.}}$, or $= 0{\cdot}9705 \frac{\text{Earth's quadrant}}{\text{Second}}$. (79).*

The resistance, or quotient of an electromotive force by a current-strength, may also be expressed as a velocity, and may actually be so physically conceived. For instance, the resistance of a straight wire of unit-length is that velocity with which it must move through a unit magnetic field, under the normal conditions (p. 201), in order to produce in it a unit-current, its ends being connected by a conductor without resistance, and which does not experience induction.

Connection of absolute Galvanic Measure with Current Work.—The advantage of Weber's system of electro-magnetic measure, which was first adopted only as the simplest ex-

* Professor Kohlrausch has since found, as the mean result of very careful experiments, $9717000000 \frac{\text{mm.}}{\text{sec.}}$ (*Trans.*)

pression of the reciprocal action of electricity and magnetism, is shown by the fact that another primary law of current action receives its simplest form by the employment of this measure. The quantity of heat A evolved by a current i in a conductor of resistance u in time t, is proportional to i^2wt, or eit, where e is the electromotive force which urges the current i through the conductor of resistance w. But by the employment of absolute measure, and by taking, further, as unit of heat, that quantity which is equivalent to unit of work (p. 192), the law takes its simplest form $A = i^2wt = eit$, as shown by Helmholtz. We may consider A the internal work of the current. It may be remarked here that as the dimensions of a current-strength are $l^{\frac{1}{2}}m^{\frac{1}{2}}t^{-1}$, and those of a resistance lt^{-1}, the product i^2wt has the dimensions l^2mt^{-2}, that is those of "work."

This rule follows from the universal law of current induction, as expressed on p. 202, in connection with that of the conservation of energy. In a closed conductor, which is moved under the influence of a magnet, an induced current is produced, which exerts a motive force on the magnet, which is always opposed to that causing the actual motion. By this motion, therefore, work is done, which is proportional to the product of the resisting force and the distance passed over. The distance is ut, where $u =$ the velocity, and t the duration of the motion; the force is always proportional to i, the strength of the induced current. We may take the force as ki, and hence have $kiut =$ the work performed.

k frequently signifies that force which will be exerted by a unit-current in the conductor under the given relations of the magnet. But as the law of induction (p. 202) asserts that ku is the electromotive force e in absolute measure, we have also for the work performed, $kiut = eit = i^2wt$.

Now since the motion produced, as effecting this work, only exists in the form of heat liberated by the current in the conductor, it follows from the law of the equivalence of work and heat, that eit or i^2wt stands for that quantity of heat into which the mechanical work is converted by means of the current; and that quantity of heat is naturally taken as unit which is equivalent to unit of work.

But necessarily the heat liberated in the conductor is due to the interior action of the current, and hence we have in i^2wt or eit the amount of heat liberated by a current i when it traverses a

conductor of resistance w, or is produced by the electromotive force e; or, in other words, its interior work.

Let us take, for instance, the current 1 *Weber* in a conductor of resistance 1 *Siemens* $= 9705 \times 10^6 \frac{\text{mm.}}{\text{sec.}}$ in absolute measure. The internal work performed per second is here $9705 \times 10^6 \frac{\text{mm.}^2 \text{ mgr.}}{\text{sec.}^2}$, or, reduced to the unit of work practically employed (p. 192) $\frac{9705 \times 10^6}{9806 \times 10^6} = 0{\cdot}990$ gramme-metres. Since 424 gramme-metres correspond to 1 calorie (1 gramme water 1° c.), in this case the quantity of heat will be $\frac{0{\cdot}990}{424} = \frac{1}{428}$ calorie.

It is not without interest to remark here that the Siemens's unit of resistance is related to Weber's unit of current, almost exactly in the same proportion as 1 gramme-metre of work in raising a load, to the absolute unit of work $1 \frac{\text{mm.}^2 \text{ mgr.}}{\text{sec.}^2}$

We may now define the Weber's unit in relation to the unit of current in the following manner:—

> The unit of electromotive force is that force which, in producing a unit-current, performs unit-work in unit-time.

Or the

> Unit of resistance is the resistance of that conductor in which unit-current performs unit-work in unit-time.

APPENDIX B.

Determination of Wave-length of Spectral Lines by Comparison on the Refraction Spectrum.

Since spectroscopes are not only constructed with different scales and dispersive powers, but the measurements of the same instruments are variable with temperature and other causes, it becomes necessary to have some standard scale to which all observations may be reduced for comparison. Such a scale is furnished by the wave-length of the lines, but from insufficient light this can rarely be measured directly by

diffraction, as described in (41). If, however, the spectrum be compared with a sufficient number of rays of which the wave-length is known (Tables 19*a*, 19*b*), that of the unknown intermediate rays may easily be determined by comparing the two scales as described in (40, 4). If, for instance, the positions of the spectral lines to be determined and a sufficient number of known lines have been laid down on the arbitrary scale of the spectroscope, the two are mapped together along the lower edge of a sheet of paper.* The known lines are also laid down on a wave-length scale down one side of the sheet. Perpendiculars are then erected on the known lines of both scales, and a curve drawn through the points where these intersect. If, now, any line on the spectroscope scale be produced to cut this curve, and a perpendicular dropped from the point of intersection to the wave-length scale, the point where it cuts the latter will give the position of the new line with regard to the wave-length.

If this method be carried out on a sufficiently large scale, it is not only the most convenient, but one of the most exact. When, however, it is only necessary to determine the positions of one or two lines instead of those of a whole spectrum, the following interpolation formula (W. Gibb's, *Silliman's Journal*, 1870) may be more convenient. If λ_1 and λ_2 are the wave-lengths of two known lines, n_1 and n_3 their positions on the spectroscope scale, or by the readings of the micrometer eye-piece, and n_2 that of an intermediate line, wave-length λ_2 of the latter will be—

$$\lambda_2^2 = \frac{n_3 - n_1}{\dfrac{n_2 - n_1}{\lambda_3^2} + \dfrac{n_3 - n_2}{\lambda_1^2}}$$

The following example from Dr. Watt's "Index of Spectra" (in which the wave-lengths of almost all known lines are given) will make the use of this formula clear. "One of the brightest lines in the spectrum of the Bessemer flame falls between two bright lines produced by cadmium. Reference to the table shows that these lines have wave-lengths 5378 and 5337 respectively.

* Suitable paper, ruled in inches and tenths, may be obtained of Messrs. Letts and Co.

When the cross-wires of the telescope were made to coincide with the lines, the micrometer-screw of the instrument gave the readings 14·38 and 15·27, while, when the wires were brought on the Bessemer line, the reading was 14·81. Putting then $n_3 = 15·27$, $\lambda_3 = 5327$, $n_1 = 14·38$, $\lambda_1 = 5378$, and $n_2 = 1481$, we find for λ_2 the value 5358."

If the line lies, not between the two reference lines, but on the less refrangible side of them—

$$\lambda_1^2 = \frac{n_3 - n_2}{\frac{n_3 - n_1}{\lambda_2^2} - \frac{n_2 - n_1}{\lambda_3^2}},$$

and if more refrangible—

$$\lambda_3^2 = \frac{n_2 - n_1}{\frac{n_3 - n_1}{\lambda_2^2} - \frac{n_3 - n_2}{\lambda_1^2}}.$$

These formulæ are troublesome to work, as logarithms are almost necessary, and yet cannot be used throughout.

Professor A. S. Herschell has shown that in spectroscopes with fixed prisms—such, for instance, as Browning's direct-vision instruments—the dispersion is nearly proportional to the inverse fourth powers of the wave-lengths.

We may therefore obtain the inverse fourth power of the wave-length of any line from that of two lines for which it is known by simple rule of three:—

$$\lambda_2^{-4} - \lambda_1^{-4} : \lambda_3^{-4} - \lambda_1^{-4} :: n_2 - n_1 : n_3 - n_1.$$

Herschell gives the following table, which is sufficiently extensive for the small direct-vision instruments to which the method was originally applied, but, of course, by employing more reference lines any required exactness may be obtained.

Fraunhöfer Lines.	Inverse fourth power of wave-length 1 mm. as unit. Billions.	Differences.
B	4·454	
		0·969
C	5·423	
		2·870
D	8·293	
		4·672
E	12·965	
		5·087
F	18·052	
		11·401
G	29·453	
		12·169
H	41·622	

Note.—The spectra of small direct-vision instruments (like Browning's "Miniature") are conveniently measured by a scale fixed at a suitable distance at one side of the instrument, and visible at the same time as the spectrum by direct reflection on the oblique face of the prism, through a small hole drilled in the tube of the eye-piece. (See *Nature*, October 10, 1872; and *Proceedings Newcastle Chem. Soc.*, December 1872, and January 1873.)

APPENDIX C.

Measurement of Potential by Thomson's Electrometers.

These instruments are of two very distinct forms—the "portable" and the "quadrant" electrometers. The action of each depends on measurement of the attraction between two planes, one of which is electrified to a constant potential, and the other brought to that which is to be measured. In both forms the potential of the electrified plane is kept tolerably constant by being connected with a Leyden jar of considerable capacity formed by the case of the instrument.

The Portable Electrometer consists of a light disc or "trap-door" of aluminium, balanced at the end of a lever, and connected with the electrified interior of the glass case. Opposed to this is a larger disc, which is connected with the object of which the potential is to be measured, and which is movable towards or from the electrified "trap-door" by a micrometer-screw. The trap-door is, of course, attracted by the disc, and must be brought to a constant position indicated by a "sight" on the end of its lever, by varying the distance of the disc. If, when the jar is negatively charged, and the disc connected with earth, the reading of the micrometer-screw be D, and when the disc is connected with a body of potential V, the reading be D',—

$$V = (D' - D)\,c,$$

in which c is a constant to be determined for each instrument. If V be negative, $(D' - D)$ will also be $-$. If, then, one electrode of a battery be put to earth, and the other connected with the disc of the electrometer, the electromotive

force of the battery may be directly measured in electrostatic units. The ordinary portable instrument will measure a difference of potential not less than that of about 1 Daniell's cell.

The quadrant electrometer is much more sensitive, and will indicate a difference of tension of $\frac{1}{100}$th that of a Daniell's cell. It consists of an aluminium plane or "needle," suspended in a sort of box divided into four quadrants, of which the opposite pairs are connected with each other, and with electrodes on the exterior of the instrument. The whole is contained in an inverted glass shade, which contains sulphuric acid to dry the air, and which serves as a Leyden jar to keep the needle at a constant potential, the latter being connected with it by a platinum wire dipping in the acid. The needle is suspended so that it is equally within each of the four quadrants, and is retained in its position either by a bifilar suspension, or by a small magnetic needle attached to it, and acted on by permanent magnets outside. Its motion is measured by attached mirror and scale (47). When all the quadrants are at the same potential the needle is equally attracted, and is not moved. If, however, one pair be at higher potential than the other, the needle is turned till the attraction is balanced by the torsion of the threads, or the directive force of the magnets.

If V be the potential of the needle, and V_1 and V_2 those of the pairs of opposite quadrants, moderate deflections are proportional to $(V_1 - V_2)\left[V - \frac{V_1 + V_2}{2}\right]$. The greater the potential of the needle, and the more sensitive the instrument, and where $\frac{V_1 + V_2}{2}$ is very small compared with V, the deflections will be nearly proportional to the difference of potentials of the quadrants multiplied by that of the needle. For quantitive measurements the latter must be kept by a "replenisher" at a constant value indicated by an attached "gauge-electrometer," and the value of the scale determined either by comparison with a standard electrometer, or by a known electromotive force, as that of a Daniell's cell.

(See *Brit. Assoc. Rep.* 1867, p. 489; J. C. Maxwell, vol. i. par. 218; Everett's *Deschanel's Physics*, par. 469B; Latimer Clark, p. 111.)

APPENDIX D.

CONDENSERS OR ACCUMULATORS.

The electromotive force, or potential of a battery, may be determined by measuring the quantity of the charge which it can give to a Leyden jar, or analogous arrangement. As an accumulator of great capacity is required to receive a perceptible charge from a feeble electromotor like a galvanic battery, it usually consists of a large number of sheets of tin-foil, alternately connected with opposite electrodes, and separated by mica or paraffined paper.

The charge Q received by a condenser is the product of its electrostatic capacity C into the electromotive force E, by which it is charged, or $Q = EC$. This quantity may be measured by discharging the condenser through a sensitive galvanometer, when the transient current will act like a sudden blow on the magnet, causing it to swing to an extreme deflection θ. Then n being the reduction-factor of the galvanometer (66), and t the time of a single vibration of its needle, the quantity Q of the discharge is—

$$Q = \frac{2nt}{\pi} \sin \tfrac{1}{2} \theta = EC.^{*} \qquad (1.)$$

From Q we may obtain the electromotive force of the battery if we know the capacity of the condenser, or *vice versa*.

To determine C in absolute measure we may determine the resistance R through which the battery employed to charge the condenser will produce unit-deflection of the galvanometer. (R, of course, includes the internal resistance of the battery and that of the galvanometer, but as R is very large these may sometimes be neglected.)

* This formula assumes no perceptible resistance to the swing of the needle, which may be the case if its moment of inertia be considerable. If this is not the case we may write, if the log. decrement λ (50) be small—

$$Q = \frac{nt}{\pi} (1 + \tfrac{1}{2}\lambda)\, \theta.$$

Then (68, III.; 74, I.), with unit deflection—

$$E = Rn, \tag{2.}$$

And combining this with equation (1), we obtain—

$$C = \frac{2t}{\pi R} \sin \tfrac{1}{2} \theta.$$

With a proper commutator-key the methods of reversal (76, I.) or of multiplication (76, II.) may be employed, but the latter is not to be recommended where great accuracy is required. From the electric "absorption" of the condenser (residual charge), the time of electrification and that of contact during discharge through the galvanometer considerably influence the result. (See J. Clark Maxwell, 771, *et seq.*; Latimer Clark, p. 120; *Brit. Assoc. Reports*, 1863, p. 144; 1867, p. 484.)

If by means of a commutator or *wippe* the electrode of a condenser can be connected alternately with the opposite poles of a battery, at short but regular intervals of time T, the condenser will be discharged and recharged in the opposite way at each reversal, and a regular series of currents will be produced, which will act on a galvanometer like a constant current, if T be small compared to the time of one vibration of the needle. The strength of this current will be $\frac{2EC}{T}$, where C is the capacity of the condenser; and if the same battery gives an equal current through a certain resistance R, $\frac{E}{R} = \frac{2EC}{T}$, and $R = \frac{T}{2C}$, so that the condenser and its commutator may be directly compared to a resistance, and indeed may be substituted for it in a Wheatstone's bridge, or with the differential galvanometer. (J. Clark Maxwell, 775.)

APPENDIX E.

Additional Methods of Measurement of Resistance

1. *Comparison of very great Resistances.* (J. C. Maxwell, 353.)

I. By difference of potential of the two ends. If a

current be passed through a conductor, the difference of potential between its ends is proportional to its resistance. If the resistances are great, this may be measured by the quadrant electrometer.

$$—P\overset{R_1}{———}P_1\overset{R_2}{———}P_2\overset{R_3}{———}P_3—$$

The resistances R_1 R_2 R_3, etc., are arranged in a series, and the current from a battery of great electromotive force is sent through them. If $(P-P_1)$, (P_1-P_2), etc., be the differences of potential corresponding to the resistances—

$$R_1 : R_2 : R_3 = (P-P_1) : (P_1-P_2) : (P_2-P_3).$$

II. Four large resistances may be arranged as a Wheatstone's bridge, and an electrometer substituted for the galvanometer. This has the advantage over the galvanometer of allowing no current to pass through it.

III. If the resistance be so great that no current measurable by a galvanometer can be forced through it, the quantity of electricity which passes in a certain time may be accumulated in a condenser, and measured by discharge through a galvanometer (p. 210). This quantity is inversely proportional to the resistance, if the electromotive force be constant. (Bright and Clark's method for leakage of joints of cables.)

IV. A condenser of great capacity is charged and allowed gradually to discharge itself through the great resistance to be measured, while the difference of potential between its surfaces is measured by an electrometer. Calling C the capacity of the condenser, E_0 the original reading of the electrometer (which may be in arbitrary units), and E that after time t, we have—

$$R = \frac{t}{C\,(log_e E_0 - log_e E)}.$$

As the condenser itself is not perfectly insulated, two experiments must be made, in one of which the condenser is dis-

charged by its own leakage alone. If R_0 be the resistance of the condenser alone, and R' that of the condenser and conductor conjointly; then the resistance R of the latter is given by the equation—

$$\frac{1}{R}=\frac{1}{R'}-\frac{1}{R_0}.$$

V. This method is much used in testing the insulation of telegraphic cables. The cable itself is used as the condenser, and as both its capacity and the area of its coating increase in the same ratio with its length, the resistance of the coating per unit of surface is simply proportional to the time in which it loses a definite portion (usually $\frac{1}{2}$) of its charge. In practice the electrometer is generally employed in conjunction with Thomson's slide resistance, which consists of a series of 100 resistance coils of 100 Ohms each. Each of the junctions between the coils is connected to one of a series of metal blocks, which are traversed by a sliding contact. One end of the series is put to earth, and the other connected with one pole of a battery, so that the potential rises uniformly through the series from 0 to that of the battery end, which we may call 100.

One electrode of the electrometer is connected with the cable, and another with a loose wire, which is shifted from block to block of the slide as the potential falls, so as to keep the electrometer at zero. The cable is charged from the battery end of the slide, and the time noted at which its potential is equal to that of each successive block. If E be the original potential, and e that to which it has fallen in time t, it will fall to $\frac{1}{2}E$ in time $T=\frac{t\, log_{10} 2}{log_{10} \frac{E}{e}}$. (Latimer Clark, pp. 111, 117.)

2. *Thomson's Method of Determining the Resistance of a Galvanometer by its own Deflection.* (*Proceedings Royal Society*, Jan. 19, 1871; J. C. Maxwell, par. 356.)

If in Fig. 14 (p. 165) we substitute the galvanoscope

itself for one of the resistances, say R, no current will pass through the branch G when $a : b = R : W$, and this is known to be the case when opening or closing a key in the branch G produces no alteration in the deflection of the galvanoscope. Then the resistance of the galvanoscope $R = \frac{Wa}{b}$.

3. *Mance's Method of Determining the Internal Resistance of a Battery.* (*Proceedings Royal Society*, Jan. 19, 1871. J. C. Maxwell, par. 357.)

The arrangement is the same as the last, except that battery and galvanometer change places—that is, in Fig. 14 R stands for the battery, and E for the galvanometer. The resistances are varied till closing the circuit at G does not alter the deflection of E; when the resistance of the battery is—

$$R = \frac{Wa}{b}.$$

If i be the current in E, the electromotive force of the battery is—

$$e = i\left(r + W + \frac{W}{b}(r + a)\right).$$

As the galvanometer is most sensitive when its deflection is small, we may bring the needle nearly to zero by a permanent magnet placed near it, before making contact at G.

APPENDIX F.

Determination of Electromotive Force by Clark's Potentiometer.

Of the methods described in (73), I. and II. are merely approximate, while even III. and IV. have the disadvantage that one of the batteries which are being compared must be in action, and consequently that any slight variation in its resistance will affect the result. In Clark's method both batteries are inactive, and this disadvantage is avoided.

Let E be a battery of greater electromotive force than either of those to be compared. If its poles be united by a wire of uniform resistance ab, the potential will fall continuously from b to a. If, then, we connect the similar pole of a standard cell e_1 with a, its potential is equal to that of a, and it is obvious that at some point b_1 in ab the potential will be equal to that of the other pole of e_1 if the resistance of ab be sufficient, and if they be connected no current will flow. This is indicated by a galvanoscope G_1, included in the circuit. If, now, another cell be similarly connected with a, a point b_2 will be found, at which its potential also will be balanced by that of ab. But since the potential falls uniformly from a to b, and, when no current passes, the difference of potentials of the poles measures the electromotive force of a battery, the electromotive forces of e_1 and e_2 will be proportional to the resistances ab_1 and ab_2, or, measuring resistances from a—

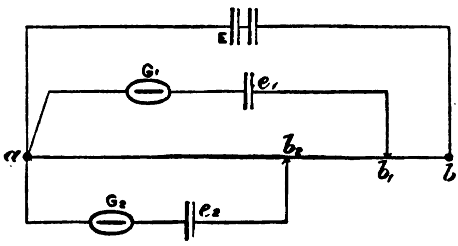


Fig. 25.

$$e_1 : e_2 :: b_1 : b_2$$

A practically convenient arrangement is to make ab_1 a stretched fine wire like that of the Wheatstone bridge (70, 2), and to bring the current in G_1 to zero by intercalating resistance in the conductor $Eb\,b_1$. If, then, ab_1 be suitably divided into a scale of say 100 parts, and the standard cell be equal to 1 volt. (see Table 24*c*), the scale will give the required electromotive force e_2 at once in absolute measure. (Latimer Clark, *Elec. Measurement,* p. 106; J. Clark Maxwell, par. 358.)

It is obvious that the principle of the potentiometer is identical with that of Thomson's slide resistance, and that a sufficiently sensitive electrometer might be substituted for either of the galvanoscopes, as described in Appendix E, (V.)

Clark employs a platinum wire coiled on an ebonite cylinder, instead of a simple stretched wire.

APPENDIX G.

Application of Electrical Resistance to Measurement of Temperature.

The resistance of metals increases with temperature, that of the pure metals much more rapidly than that of alloys; and it is for this reason that standard resistance-coils are made of platinum-silver, gold-silver, or german-silver, in preference to the pure metals. Dr. C. W. Siemens has taken advantage of this variation of resistance for thermometric and pyrometric purposes.

In measuring resistances below 100° two coils may be employed, one of which, the thermometrical coil, may be buried, sunk in the sea, or placed wherever it is desired to measure the temperature, being carefully protected against moisture; while the second or comparison coil is plunged in a water bath, of which the temperature may be raised or lowered, and accurately measured with a mercury thermometer. The two coils are connected by a light cable, containing three insulated wires. The current from a battery flows down one of these, and is then divided, one portion passing through the thermometer coil, and then back through one of the remaining wires, while the other portion passes directly back through the third wire, and then through the comparison coil, both currents being finally led back to the battery through a differential galvanometer (69, II.) Both conducting wires are thus under the same temperature conditions, and the coils alone can vary independently. When they are at the same temperature their resistances are equal, and the galvanometer is undeflected. The temperature of the comparison coil is varied till this is the case, and its temperature is then that of the thermometer coil. The comparison coil may be dispensed with, and the temperature determined directly by calculation from the measured resistance

of the thermometer coil. Taking T as the "absolute" temperature, reckoned from $-273°$ C., the resistance r is for—

Platinum	$r = 0{\cdot}039369T^{\frac{1}{2}} + 0{\cdot}00216407T - 0{\cdot}2413$
Copper	$r = 0{\cdot}026577T^{\frac{1}{2}} + 0{\cdot}0031443T - 0{\cdot}22751$
Iron	$r = 0{\cdot}072545T^{\frac{1}{2}} + 0{\cdot}0038133T - 1{\cdot}23971$

And Dr. Siemens has proved that this holds good for platinum through a range of 1000°.

For low temperatures the coil is generally of iron or copper, but for those of furnaces, etc., it consists of a platinum wire coiled on a small porcelain cylinder, and protected by a closed iron or platinum tube. If the temperature of the furnace does not exceed a full red heat, the coil may be left in it continuously; if hotter it must be exposed only for a short measured interval, say three minutes, and will acquire a temperature lower than that of the furnace by a determinable small amount.

A Wheatstone bridge may, of course, be substituted for the differential galvanometer, and Siemens had also employed a differential voltameter, consisting of two similar voltameter tubes. In this case the polarisation and resistance of both are similar, and counterbalance and eliminate each other; and if γ be the resistance of the voltameter, and C the known resistance, and C' the unknown, while V and V' are the volumes of gas liberated in a given time; $(C + \gamma)\, V = (C' + \gamma)\, V'$, and

$$C^1 = \frac{V}{V^1}(C + \gamma) - \gamma.$$

VV' may be measured in any arbitrary units, and temperature and pressure may be neglected if alike for both. The acid may be brought to the same level within and without the tubes by small supply reservoirs connected by india-rubber tubes.

By valves at the top of the tubes the gases are brought to zero at the beginning of each observation. (*Proceedings Royal Society*, April 27, 1871; and J. C. Maxwell, par. 360.)

APPENDIX H.

MEASUREMENT OF SHORT INTERVALS OF TIME.

In physical experiments it is often necessary to measure short intervals of time, when perfect and elaborate instruments like the electric chronograph are not accessible. The following methods are simple, and only require apparatus easily obtained.

I. *Pouillet's Method by Deflection of a Galvanometer.*—It is arranged that a circuit, including a constant battery and a galvanometer, shall be closed during the short time to be measured. If the period of closure be short compared to the time of oscillation of the needle, the amplitude of its first swing from its place of rest will be proportional to the time of contact. To determine the actual value in time of the deflection, M. Pouillet employed a rotating glass disc with a metal radial strip, which for a small part of its revolution made contact with a spring. M. Schneebeli (*Pogg. Ann.*, vol. cxliii. p. 239; and *Phil. Mag.* vol. xliv. p. 477), who has successfully applied the method to measure the duration of collision of elastic bodies, used a metallic pendulum, carrying at its lower part a triple spring, which rubbed on a strip of steel fixed in the same vertical plane as the axis of rotation of the pendulum. A glass plate applied horizontally to the steel caused the spring to slide into it without shock. The duration t of contact was inversely proportional to the square root of the height of fall H of the pendulum, and b being the breadth of the strip—

$$t = \frac{b}{\sqrt{2Hg}}.$$

If the resistance R and electromotive force t of the circuit be known, and consequently the quantity passing through the galvanometer in unit of time, the time t may be found by formula (1), Appendix D.

$$Q = \frac{Et}{R} = \frac{2nT}{\pi} \sin \tfrac{1}{2}\theta.$$

II. The interval of regular pulses is given by the pitch of the note produced (see Table 18). This may be determined either by a syren (Tyndall, *Sound*, p. 64), which is brought into unison with it, and which counts its pulses by a clockwork register; or by determining the length of wave in air (37). The vibrations may often also be made to record themselves on a sheet or cylinder of smoked glass, which is drawn under a point attached to the moving object, a tuning-fork of known number of vibrations per second being similarly made to register at the same time. The velocity of rotation of a disc or gyroscope may be measured by smoking the rim of the disc, and lightly touching it with a tiny cone of india-rubber attached to one prong of a vibrating tuning-fork. The lamp-black will be rubbed off in spots, and if n contacts correspond to m degrees of angular rotation of the disc, and the fork vibrates t times per second, the number of revolutions of the disc per second will be—

$$N = \frac{mt}{360n}.$$

If the primary of an induction-coil be put in circuit by a connection which is broken at a given moment, a current will be produced at the instant of rupture in the secondary, which may register itself either by piercing a revolving paper disc with its spark, or, similarly, by marking a black-varnished metallic disc, or by making a brown dot on a paper prepared with potassic iodide and starch. A smaller spark is also produced by closure. This method has been applied to measure the velocity of a bullet, which successively divided wires placed across its course. If the velocity of the disc was not known, it might, of course, be determined by employment of a second coil in which contacts were made and broken by a large tuning-fork, or other isochronous vibratior.

III. MM. Lucas and Cazin (*Phil. Mag.* vol. xl. p. 78)

describe a chronoscope employed by them to measure the duration of the electric spark, in which they avail themselves of the principle of the vernier. A blackened mica disc of 15 cm. diameter was divided round the edge into 180 transparent divisions, and rotated 100 to 300 times per second. A silvered glass disc was fixed very near to it, which had six transparent divisions like those of a vernier. The light of the spark was observed with a telescope through the transparent divisions, and the number of these which could be seen by one spark was counted.

IV. *Wheatstone's Method* for time between two sparks in the same conductor consisted in observing the angular deviation between the two images reflected in a mirror rotating on an axis parallel to its plane. In some later observations this deviation has been measured by making the two images coincide by means of a telescope with divided object-glass, of which one-half was movable by a micrometer-screw.

APPENDIX I.

Resistance of Mercury in Tubes.

If l be the length of the glass tube, g the weight of mercury it contains, and σ its sp. gr. ($=13{\cdot}557$), its resistance W in Siemens's units at 0°C. is

$$W=\frac{l^2\sigma}{g}.$$

The resistance of mercury increases 8·3 op. between 0° and 100°C. Glass tubes are always conical, for which we must correct by multiplication by $\dfrac{1+\sqrt{a}+\dfrac{1}{\sqrt{a}}}{3}$, where a is the ratio between the greatest and smallest area of section, which varies inversely with the length of a short column of mercury in different parts of the tube (see Art. 23).

The connections must be made by stout amalgamated copper electrodes pressed against the ends of the tube (Compare *Brit. Ass. Rep.* 1862.)

TABLES.

TABLE 1.

Density of Bodies

(a) *Solid and Fluid Bodies.*

Aluminium	2·6	Wood, beech	0·75	Ice at 0°	0·9167
Lead	11·3	„ oak	0·65	Water at 4°	1·000
Bronze	8·6	„ pine	0·5	Ether at 15°	0·7202
Iron, malleable	7·75	Copper	8·9	Alcohol at 15°	0·7938
„ cast	7·5	Brass	8·5	Olive oil	0·915
„ wire	7·65	German Silver	8·5	Mercury at 0°	13·596
Cast steel	7·8	Platinum	21·5	Nitric Acid— concentd. at 15°	1·52
Ivory	1·9	Silver	10·4		
Glass, crown	2·7	Wax	0·96	Sulphuric Acid— concentd. at 15°	1·843
„ flint	3·5	Zinc	7·1		
Gold	19·3	Tin	7·3	Spirits of Turpentine	0·872

(b) *Gaseous Bodies.*

	At 0° temp. and 760 mm. pressure compared to water.	Compared to air at similar pressure and temperature.
Air	0·0012928	1·00000
Oxygen	0·0014293	1·10563
Nitrogen	0·0012557	0·97137
Hydrogen	0·00008954	0·06926
Carbonic dioxide	0·0019767	1·52910
Mixed gases from electrolysis of water	0·0005361	0·41472
Aqueous vapour	...	0·6230

TABLE 2.

Reduction of Arbitrary Hydrometer Scales.

Lighter than Water.				Heavier than Water.			
Sp. gr.	Baumé.	Beck.	Cartier.	Sp. gr.	Baumé.	Beck.	Twaddell.
0·75	58°·4	56°·7		1·0	0°·0	0°·0	0°·0
0·80	46·3	42·5	43·0	1·1	13·2	15·4	20·0
0·85	35·6	30·0	33·6	1·2	24·2	28·3	40·0
0·90	26·1	18·9	25·2	1·3	33·5	39·2	60·0
0·95	17·7	8·9	17·7	1·4	41·5	48·6	80·0
1·00	10·0	0·0	11·0	1·5	48·4	56·7	100·0
				1·6	54·4	63·7	120·0
				1·7	59·8	70·0	140·0
				1·8	64·5	75·6	160·0
				1·9	68·6		180·0
				2·0	72·6		200·0

TABLE 3.

Percentage Contents and Specific Gravity of Aqueous Solutions of Sulphuric, Nitric, and Hydrochloric Acids, Common Salt, Caustic Potash and Soda, Ammonia, Sugar Candy, and Alcohol, at 15° C.

The percentage contents is percentage weight, except in the case of Alcohol, where it is percentage volume of absolute Alcohol.

Percentage Contents.	Specific Gravity.									Percentage Contents.
	So_3Ho_2.	N_2O_5.	HA.	NaCl.	KHo.	NaHo.	NH_3.	Sugar.	Alcohol.	
0	0·999	0·999	0·999	0·999	1·00	1·00	0·999	0·999	0·999	0
5	1·032	1·034	1·024	1·036	1·06	1·06	1·979	1·019	0·992	5
10	1·068	1·068	1·047	1·073	1·11	1·13	0·959	1·039	0·986	10
15	1·106	1·104	1·073	1·111	1·17	1·22	0·941	1·061	0·980	15
20	1·144	1·141	1·098	1·150	1·23	1·30	0·925	1·083	0·975	20
25	1·182	1·179	1·123	1·191	1·30	1·35	0·911	1·106	0.970	25
30	1·223	1·219	1·149	...	1·37	1·42	0·898	1·129	0·965	30
35	1·264	1·257	1·174	...	1·43	1·48	0·886	1·154	0·958	35
40	1·306	1·296	1·197	...	1·48	1·54	...	1·179	0·951	40
45	1·351	1·330	...	...	1·57	1·61	...	1·205	0·943	45
50	1·398	1·362	...	...	1·66	1·67	...	1·233	0·933	50
55	1·448	1·394	...	...	...	1·73	...	1·261	0·923	55
60	1·501	1·420	...	...	...	1·80	...	1·289	0·913	60
65	1·557	1·444	...	...	...	1·87	...	1·318	0·901	65
70	1·615	1·467	...	...	...	1·92	...	1·347	0·889	70
75	1·675	1·485	...	...	...	1·96	...	...	0·876	75
80	1·734	1·500	...	...	...	...	...	...	0·863	80
85	1·786	...	...	...	...	...	...	...	0·849	85
90	1·822	...	...	...	...	...	...	...	0·833	90
95	1·838	...	...	...	...	...	...	...	0·816	95
100	1·843	...	...	...	...	...	...	...	0·794	100

TABLE 4.

Density Q of Water at Temperature $t°$.

(From determinations of Halturön, Jolly, Kopp, Matthiessen, and Pierre.)

t.	Q.	Diff.
0°	0·99988	
1	0·99993	−5
2	0·99997	−4
3	0·99999	−2
4	1·00000	−1
5	0·99999	+1
6	0·99997	2
7	0·99994	3
8	0·99988	6
9	0·99982	6
10	0·99974	8
11	0·99965	9
12	0·99955	10
13	0·99943	12
14	0·99930	13
15	0·99915	15
16	0·99900	15
17	0·99884	16
18	0·99866	18
19	0·99847	19
20	0·99827	20
21	0·99806	21
22	0·99785	21
23	0·99762	23
24	0·99738	24
25	0·99714	24
26	0·99689	25
27	0·99662	27
28	0·99635	27
29	0·99607	28
30	0·99579	28

TABLE 5.

Expansion of Water from 0° to 100°.

Volume of 1 Gramme of Water in Cubic Centimetres.

Temp.	Volume.	Increase per 1°.
0°	1·0001	
4	1·0000	
10	1·0003	
15	1·0009	0·00012
20	1·0017	0·00016
25	1·0029	0·00024
30	1·0043	0·00028
35	1·0059	0·00032
40	1·0077	0·00036
45	1·0097	0·00040
50	1·0120	0·00046
55	1·0144	0·00048
60	1·0170	0·00052
65	1·0197	0·00054
70	1·0227	0·00060
75	1·0258	0·00062
80	1·0290	0·00064
85	1·0323	0·00066
90	1·0358	0·00070
95	1·0395	0·00074
100	1·0432	0·00074

TABLE 6.

Density of Dry Atmospheric Air

Compared to Water at 4° C.

For Temperature *t* and Barometric Pressure *b* (in Lat. 45°).

(By R. Kohlrausch from Regnault's Observations.)

t.	*b* = 720mm.	730mm.	740mm.	750mm.	760mm.	770mm.	Prop. Parts.	
	0·00	0·00	0·00	0·00	0·00	0·00		
0°	1225	1242	1259	1276	1293	1310	17	
1	1220	1237	1254	1271	1288	1305	1mm.	2
2	1216	1233	1249	1267	1283	1300	2	3
3	1212	1228	1245	1262	1279	1296	3	5
4	1207	1224	1241	1257	1274	1290	4	7
5	1203	1219	1236	1253	1270	1286	5	8
6	1198	1215	1232	1248	1265	1282	6	10
7	1194	1211	1227	1244	1260	1277	7	12
8	1190	1206	1223	1239	1256	1272	8	14
9	1186	1202	1219	1235	1251	1268	9	15
10°	1181	1198	1214	1231	1247	1263	16	
11	1177	1194	1210	1226	1243	1259	1mm.	2
12	1173	1189	1206	1222	1238	1255	2	3
13	1169	1185	1202	1218	1234	1250	3	5
14	1165	1181	1197	1214	1230	1246	4	6
15	1161	1177	1193	1209	1225	1242	5	8
16	1157	1173	1189	1205	1221	1237	6	10
17	1153	1169	1185	1201	1217	1233	7	11
18	1149	1165	1181	1197	1213	1229	8	13
19	1145	1161	1177	1193	1209	1224	9	14
20°	1141	1157	1173	1189	1204	1220	15	
21	1137	1153	1169	1185	1200	1216	1mm.	1
22	1133	1149	1165	1181	1196	1212	2	3
23	1130	1145	1161	1177	1192	1208	3	4
24	1126	1141	1157	1173	1188	1204	4	6
25	1122	1138	1153	1169	1184	1200	5	7
26	1118	1134	1149	1165	1180	1196	6	9
27	1114	1130	1145	1161	1176	1192	7	10
28	1110	1126	1142	1157	1172	1188	8	12
29	1107	1122	1138	1153	1169	1184	9	13
30°	1103	1119	1134	1149	1165	1180		

TABLE 7.

Reduction of Volume of Gas to 0° C. $\alpha = 0{\cdot}003665$.

t.	$1+\alpha t$.	t.	$1+\alpha t$.	t.	$1+\alpha t$.	t.	$1+\alpha t$.	t.	$1+\alpha t$.	Prop. Parts.	
0°	1·0000	20°	1·0733	40°	1·1466	60°	1·2199	80°	1·2932		
1	1·0037	21	1·0770	41	1·1503	61	1·2236	81	1·2969	16	
2	1·0073	22	1·0806	42	1·1539	62	1·2272	82	1·3005	1	2
3	1·0110	23	1·0843	43	1·1576	63	1·2309	83	1·3042	2	3
4	1·0147	24	1·0880	44	1·1613	64	1·2346	84	1·3079	3	5
5	1·0183	25	1·0916	45	1·1649	65	1·2382	85	1·3115	4	6
6	1·0220	26	1·0953	46	1·1686	66	1·2419	86	1·3152	5	8
7	1·0257	27	1·0990	47	1·1723	67	1·2456	87	1·3189	6	10
8	1·0293	28	1·1026	48	1·1759	68	1·2492	88	1·3225	7	11
9	1·0330	29	1·1063	49	1·1796	69	1·2529	89	1·3262	8	13
										9	14
10°	1·0366	30°	1·1099	50°	1·1832	70°	1·2565	90°	1·3298		
11	1·0403	31	1·1136	51	1·1869	71	1·2602	91	1·3335	17	
12	1·0440	32	1·1173	52	1·1906	72	1·2639	92	1·3372	1	2
13	1·0476	33	1·1209	53	1·1942	73	1·2675	93	1·3408	2	3
14	1·0513	34	1·1246	54	1·1979	74	1·2712	94	1·3445	3	5
15	1·0550	35	1·1283	55	1·2016	75	1·2749	95	1·3482	4	7
16	1·0586	36	1·1319	56	1·2052	76	1·2785	96	1·3518	5	8
17	1·0623	37	1·1356	57	1·2089	77	1·2822	97	1·3555	6	10
18	1·0660	38	1·1393	58	1·2126	78	1·2859	98	1·3592	7	12
19	1·0696	39	1·1429	59	1·2162	79	1·2895	99	1·3628	8	14
20°	1·0733	40°	1·1466	60°	1·2199	80°	1·2932	100°	1·3665	9	15

TABLE 7a.

Reduction of Volume of Gas to 760 mm. Pressure.

p.	$\frac{p}{760}$.	p.	$\frac{p}{760}$.	p.	$\frac{p}{760}$.	p.	$\frac{p}{760}$.
mm.		mm.		mm.		mm.	
700	0·9211	720	0·9474	740	0·9737	760	1·0000
701	0·9224	721	0·9487	741	0·9750	761	1·0013
702	0·9237	722	0·9500	742	0·9763	762	1·0026
703	0·9250	723	0·9513	743	0·9776	763	1·0039
704	0·9263	724	0·9526	744	0·9789	764	1·0053
705	0·9276	725	0·9539	745	0·9803	765	1·0066
706	0·9289	726	0·9553	746	0·9816	766	1·0079
707	0·9303	727	0·9566	747	0·9829	767	1·0092
708	0·9316	728	0·9579	748	0·9842	768	1·0105
709	0·9329	729	0·9592	749	0·9855	769	1·0118
710	0·9342	730	0·9605	750	0·9868	770	1·0132
711	0·9355	731	0·9618	751	0·9882	771	1·0145
712	0·9368	732	0·9632	752	0·9895	772	1·0158
713	0·9382	733	0·9645	753	0·9908	773	1·0171
714	0·9395	734	0·9658	754	0·9921	774	1·0184
715	0·9408	735	0·9671	755	0·9934	775	1·0197
716	0·9421	736	0·9684	756	0·9947	776	1·0211
717	0·9434	737	0·9697	757	0·9961	777	1·0224
718	0·9447	738	0·9710	758	0·9974	778	1·0337
719	0·9461	739	0·9724	759	0·9987	779	1·0450
720	0·9474	740	0·9737	760	1·0000	780	1·0563

TABLE 8.

Reduction of a Weighing with Brass Weights to Weight in Vacuo.

Δ.	K.	Δ.	K.
0·7	+ 1·57	2·0	+ 0·46
0·8	+ 1·36	3·0	+ 0·26
0·9	+ 1·19	4·0	+ 0·16
1·0	+ 1·06	5·0	+ 0·10
1·1	+ 0·95	6·0	+ 0·06
1·2	+ 0·86	7·0	+ 0·03
1·3	+ 0·78	8·0	+ 0·01
1·4	+ 0·71	9·0	− 0·01
1·5	+ 0·66	10·0	− 0·02
1·6	+ 0·61	12·0	− 0·04
1·7	+ 0·56	14·0	− 0·06
1·8	+ 0·52	16·0	− 0·07
1·9	+ 0·49	18·0	− 0·08
2·0	+ 0·46	20·0	− 0·08

$$\frac{k}{1000} = 0{\cdot}0012\left(\frac{1}{\Delta} - \frac{1}{8{\cdot}4}\right).$$ Compare p. 30.

If the weighed body has the density Δ, and its weight in air be m grammes, mk mgrm. must be added to reduce the weighing to vacuo.

TABLE 9.

Coefficients of Expansion for 1° C.

The length L of a body is increased by βL for each degree of increased temperature, and its volume V by $3\beta V$.

	β		β
Lead	0·0000285	Brass	0·000019
Iron	0·000012	Platinum	0·000009
Glass	0·0000085	Silver	0·000019
Gold	0·000015	Zinc	0·000029
Copper	0·0000175	Tin	0·000022

The volume V of quicksilver increases $0{\cdot}0001815V$ for 1°.

TABLE 10.

Boiling Temperature of Water t at Barometer Pressure b (after Regnault).

b.	t.	b.	t.	b.	t.	b.	t.	b.	t.
680	96°·92	700	97°·72	720	98°·49	740	99°·26	760	100°·00
681	·96	01	·75	21	·53	41	·29	61	·04
682	97 ·00	02	·79	22	·57	42	·33	62	·07
683	·04	03	·83	23	·61	43	·37	63	·11
684	·08	04	·87	24	·65	44	·41	64	·15
685	·12	05	·91	25	·69	45	·44	65	·18
686	·16	06	·95	26	·72	46	·48	66	·22
687	·20	07	97 ·99	27	·76	47	·52	67	·26
688	·24	08	98 ·03	28	·80	48	·56	68	·29
689	·28	09	·07	29	·84	49	·59	69	·33
690	·32	710	·11	730	·88	750	·63	770	·36
691	·36	11	·15	31	·92	51	·67	71	·40
692	·40	12	·19	32	·95	52	·70	72	·44
693	·44	13	·22	33	98 ·99	53	·74	73	·47
694	·48	14	·26	34	99 ·03	54	·78	74	·51
695	·52	15	·30	35	·07	55	·82	75	·55
696	·56	16	·34	36	·11	56	·85	76	·58
697	·60	17	·38	37	·14	57	·89	77	·62
698	·64	18	·42	38	·18	58	·93	78	·65
699	·68	19	·46	39	·22	59	·96	79	·69
700	97°·72	720	98°·49	740	99°·26	760	100°·00	780	100°·72

TABLE 11.

Tension of Aqueous Vapour.

In mm. of Quicksilver between 90° and 101° (Regnault).

	90°	91°	92°	93°	94°	95°	96°	97°	98°	99°	100°
	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
·0	525·4	545·8	566·8	588·4	610·7	633·8	657·5	682·0	707·3	733·2	760·0
·1	527·4	547·8	568·9	590·6	613·0	636·1	659·9	684·5	709·8	735·8	762·7
·2	529·5	549·9	571·0	592·8	615·3	638·5	662·4	687·0	712·4	738·5	765·5
·3	531·5	552·0	573·2	595·0	617·6	640·8	664·8	689·5	715·0	741·2	768·2
·4	533·5	554·1	575·3	597·3	619·9	643·2	667·2	692·0	717·6	743·8	771·9
·5	535·5	556·2	577·5	599·5	622·2	645·6	69·7	694·6	720·1	746·5	773·7
·6	537·6	558·3	579·7	601·7	624·5	647·9	672·1	697·1	722·7	749·2	776·5
·7	539·6	560·4	581·8	604·0	626·8	650·3	674·6	699·6	725·4	751·9	779·3
·8	541·7	562·5	584·0	606·2	629·1	652·7	677·1	702·1	728·0	754·6	782·0
·9	543·7	564·6	586·2	608·5	631·4	655·1	679·5	704·7	730·6	757·3	784·8

TABLE 12.

Mean Height b of Barometer at Elevation H above the Sea-level.

Temperature of Air taken at 10° C.

H.	H.	b.	b.	H.	H.	b.	b.
metres.	Eng. feet.	mm.	inches.	metres.	Eng. feet.	mm.	inches.
0	0	760	29·92	1000	3280	674	26·53
100	328	751	29·57	1100	3608	666	26·22
200	656	742	29·21	1200	3936	658	25·90
300	984	733	28·85	1300	4265	650	25·59
400	1312	724	28·50	1400	4592	642	25·27
500	1640	716	28·19	1500	4920	635	25·00
600	1968	707	27·83	1600	5248	627	24·68
700	2296	699	27·52	1700	5577	620	24·41
800	2624	690	27·17	1800	5905	612	24·09
900	2952	682	26·85	1900	6233	605	23·82
1000	3280	674	26·53	2000	6561	598	23·54

TABLE 12a.

Reduction of Millimetres to Inches.

mm.	inches.	mm.	inches.
100	3·93708	710	27·9532
200	7·87415	720	28·3469
300	11·81124	730	28·7406
400	15·74832	740	29·1343
500	19·68539	750	29·5280
600	23·62247	760	29·9217
700	27·55955	770	30·3155
800	31·49663	780	30·7091
900	35·43371	790	31·1029
1000	39·37079	800	31·4966

TABLE 13.

For Hygrometry.

Pressure of aqueous vapour *e*, and weight of water *f*, contained in 1 cubic metre of air, with dew-point *t*; or, when at the temperature *t*, the air would be saturated with aqueous vapour.

t.	*e.*	*f.*	*t.*	*e.*	*f.*	*t.*	*e.*	*f.*	*t.*	*e.*	*f.*
	mm.	gr.		mm.	gr.		mm.	gr.		mm.	gr.
−10°	2·0	2·1	0°	4·6	4·9	10°	9·1	9·4	20°	17·4	17·2
− 9	2·2	2·4	1	4·9	5·2	11	9·8	10·0	21	18·5	18·2
− 8	2·4	2·7	2	5·3	5·6	12	10·4	10·6	22	19·7	19·3
− 7	2·6	3·0	3	5·7	6·0	13	11·1	11·3	23	20·9	20·4
− 6	2·8	3·2	4	6·1	6·4	14	11·9	12·0	24	22·2	21·5
− 5	3·1	3·5	5	6·5	6·8	15	12·7	12·8	25	23·6	22·9
− 4	3·3	3·8	6	7·0	7·3	16	13·5	13·6	26	25·0	24·2
− 3	3·6	4·1	7	7·5	7·7	17	14·4	14·5	27	26·5	25·6
− 2	3·9	4·4	8	8·0	8·1	18	15·4	15·1	28	28·1	27·0
− 1	4·2	4·6	9	8·5	8·8	19	16·3	16·2	29	29·8	28·6
− 0°	4·6	4·9	10°	9·1	9·4	20°	17·4	17·2	30°	31·6	30·1

TABLE 14.

Specific Heats.

Lead	0·0314	Ether at 17° . . .	0·516
Iron	0·114	Alcohol at 17° . .	0·615
Glass	0·19	Quicksilver . . .	0·0333
Gold	0·0324	Oil of Turpentine at 17°	0·426
Copper	0·0951	Water at 0° . . .	1·0000
Brass	0·094	„ „ 10° . . .	1·0005
Platinum	0·0324	„ „ 20° . . .	1·0012
Silver	0·0570	„ „ 30° . . .	1·0020
Zinc	0·0955	„ mean between 0° and 100°	1·0050
Tin	0·0562		

TABLE 15.

Tension of Mercurial Vapour in Mm. of Mercury (Regnault).

Temp.	Tension.
	mm.
0°	0·02
20	0·04
40	0·08
60	0·16
80	0·35
100	0·75
120	1·5
140	3·1
160	5·9
180	11·0
200	19·9
220	34·7
240	58·8
260	96·7
280	155·2
300°	242·2

TABLE 16.

Capillary Depression of Mercury in a Glass Tube (after Danger).

Diameter.	Depression of Top of Curve.	Height of Meniscus.
mm.	mm.	mm.
1	5·03	0·57
2	2·18	0·95
3	1·20	1·22
4	0·70	1·41
5	0·42	1·54
6	0·25	1·62
7	0·15	1·67
8	0·10	1·68
9	0·06	1·69
10	0·04	1·69

TABLE 17.

Modulus of Elasticity E and Breaking Strain p of some Metals when stretched at 17° C. (after Wertheim).

The numbers represent $\frac{\text{Kgr.}}{\square^{\text{mm.}}}$; that is, if a wire be employed of $1\square^{\text{mm.}}$ section, E signifies the load in kilogrammes which would be required to double its length; and p the weight in kgr., which would break it. It follows that the increment of length l of a wire of length L and section $q\ \square^{\text{mm.}}$, caused by a stretching weight of P kgr. will be $l = \frac{L}{q}\frac{P}{E}$, and a wire of $q\ \square^{\text{mm.}}$ will break with a strain of qp kgr.

Example.—An iron wire is 1000 mm. in length, and 0·8 mm. in diameter; or its section $= 0{\cdot}4^2 \times 3{\cdot}14 = 0{\cdot}50\ \square^{\text{mm.}}$. It will be stretched by a load of 5 kgr. $\frac{1000\,.\,5}{0{\cdot}5 \times 19000} = 0{\cdot}53^{\text{mm.}}$ It will be broken by a load of $0{\cdot}5 \times 61 = 30{\cdot}5$ kgr.

	E	p
Lead	1800	2·1
Iron	19000	61·
Steel	21000	70·
Gold	8100	27·
Copper	12400	40·
Brass	9000	60·
Platinum	17000	34·
Silver	7400	29·
Zinc	8700	13·
Tin	4000	2·4

TABLE 18.

Pitch and Number of Vibrations per Second of Musical Notes.

	C_{-2}.	C_{-1}.	C.	c.	c_1.	c_2.	c_3.	c_4.
C	16·35	32·70	65·41	130·8	261·7	523·3	1047	2093
C♯	17·32	34·65	69·30	138·6	277·2	554·4	1109	2218
D	18·35	36·71	73·42	146·8	293·7	587·4	1175	2350
D♯	19·44	38·89	77·79	155·6	311·2	622·3	1245	2489
E	20·60	41·20	82·41	164·8	329·7	659·3	1319	2637
F	21·82	43·65	87·31	174·6	349·2	698·5	1397	2794
F♯	23·12	46·25	92·50	185·0	370·0	740·0	1480	2960
G	24·50	49·00	98·00	196·0	392·0	784·0	1568	3136
G♯	25·95	51·91	103·8	207·6	415·3	830·6	1661	3322
A	27·50	55·00	110·0	220·0	**440·0**	880·0	1760	3520
A♯	29·13	58·27	116·5	233·1	466·2	932·3	1865	3729
B	30·86	61·73	123·5	246·9	493·9	987·7	1975	3951

TABLE 19.

Lines of the Flame-Spectra of the most important Light Metals,

according to Bunsen and Kirchhoff's scale; the sodium-line being taken as 50, and the slit having a breadth of 1 division.

The first number denotes the position of the middle of the line upon the scale, the Roman figure indicates the brightness, I being the brightest, and the third number gives the breadth of the band when it exceeds 1 scale-division, the breadth of the slit.

S signifies that the line is quite sharp and clearly defined, *s* that it is tolerably so; the remaining lines being nebulous and ill defined.

The lines most characteristic of each body are printed in thick type.

The brightness of the lines of *Ca*, *Sr*, and *Ba* is that of a constant spectrum. If the chlorides be employed, the spectra are at first much brighter. In many cases the flame-spectra are really those of compounds, the spectra of the metals themselves obtained by the electric spark being frequently entirely different, and consisting of much finer lines.

The colours of the spectrum are approximately—red to 48, yellow to 52, green to 80, blue to 120, and violet beyond.

K.	*Na.*	*Li.*	*Ca.*	*Sr.*	*Ba.*
17·5 II. *s*		**32·0** I. *S*	33·1 IV. 2	29·8 III.	
			36·7 III.	**32·1** II.	
				33·8 II.	
Faint continuous spectrum from 55 to 120	**50·0** I. *S*	45·2 IV. *s*	**41·7** I. 1·5	36·3 II.	35·2 IV. 2
			46·8 III. 2	38·6 III.	41·5 III. 3
				41·5 III.	45·6 III. *s* 1·5
			49·0 III.	**45·8** I.	
					52·1 IV.
			52·8 IV.		56·0 III. 2
			54·9 IV.		**60·8** II. *s*
			60·8 I. 1·5		66·5 III. 3
			68·0 IV. 2		**71·4** III. 3
				105·0 III. *s*	**76·8** III. 2
153·0 IV.					82·7 IV. 4
			135·0 IV. *S*		**89·3** III. 2

TABLE 19*a*.

Wave-Lengths of the Principal Lines of the Solar Spectrum in Tenth-Metres in Air at 760 mm. Pressure and 16° Temperature (Angström).

In order to obtain the wave-lengths in vacuo the numbers must be multiplied by the respective refractive indices of the rays for air at 16° C. (Watts).

			Approximate Positions on Bunsen and Kirchhoff's Scale.
A	7604	1^{-10} metre	17·5
B	6867	,,	27·6
C	6562	,,	34·0
D_1	5895	,,	50·0
D_2	5889	,,	
E	5269	,,	71·0
b_1	5183	,,	75·7
F	4861	,,	90·0
G	4307	,,	127·5
H_1	3968	,,	162·0
H_2	3933	,,	166·0

TABLE 19*b*.

Wave-Lengths of some of the Principal Bright Lines in the Spectra of the Elements, and their Approximate Positions on Bunsen and Kirchhoff's Scale.

Element.	Wave-Length.		Scale Number.	
Ka	7685	1^{-10} metre	17·5	
Lia	6705	,,	32·0	
Ha	6562	,,	24·0	
Liβ	6102	,,	45·2	
Na	5892	,,	50·0	
C	5662	,,	58·	Edge of band seen in blue of candle flame.
Tl	5348	,,	67·	
C	5170	,,	75·	Edge of band in candle flame.
Hβ	4861	,,	90·	
Sr	4607	,,	105·	
Ca	4226	,,	135·	Approximate in flame spectrum.
Hγ	4101	,,	151·	
Kβ	4080	,,	153·	Flame spectrum.

TABLE 20.

Mean Indices of Refraction, and Dispersions of several Bodies.

	Index of Refraction.	Dispersion.
Crown Glass (mean)	1·53	0·022
Flint Glass "	1·60	0·042
Water	1·336	0·0132
Alcohol	1·372	0·0133
Carbon Disulphide . . .	1·68	0·0837
Canada Balsam	1·54	
Air	1·000294	

TABLE 21.

For Reduction of Time of Oscillation to an Infinitely Small Arc.

$$k = \frac{1}{4} \sin^2 \frac{\alpha}{4} + \frac{5}{64} \sin^4 \frac{\alpha}{4}.$$

If the observed time of oscillation of a magnet or pendulum be t, with an arc of oscillation of α degrees, kt must be subtracted from the observed value in order to reduce the time to that of an infinitely small oscillation.

α	k		α	k		α	k		α	k	
0°	0·00000	0	10°	0·00048	10	20°	0·00190	20	30°	0·00428	29
1	000	2	11	058	11	21	210	20	31	457	30
2	002	2	12	069	11	22	230	21	32	487	31
3	004	4	13	080	13	23	251	23	33	518	32
4	008	4	14	093	14	24	274	23	34	550	33
5	012	5	15	107	15	25	297	25	35	583	33
6	017	6	16	122	16	26	322	25	36	616	35
7	023	7	17	138	16	27	347	26	37	651	35
8	030	9	18	154	18	28	373	27	38	686	37
9	039	9	19	172	18	29	400	28	39	723	38
10°	0·00048		20°	0·00190		30°	0·00428		40°	0·00761	

TABLE 22.

Horizontal Intensity of Terrestrial Magnetism,

For Central Europe, at the beginning of the year 1870 (after Lamont's Maps from the new Göttingen Observations).

The Horizontal Intensity increases about 0·004 per year.

North Latitude.	Longitude East from Ferro.				
	20°	25°	30°	35°	40°
45°	2·06	2·09	2·14	2·18	2·22
46	2·02	2·05	2·10	2·14	2·18
47	1·98	2·01	2·06	2·09	2·14
48	1·94	1·97	2·01	2·05	2·10
49	1·90	1·93	1·97	2·01	2·05
50	1·85	1·89	1·93	1·97	2·01
51	1·82	1·85	1·89	1·93	1·97
52	1·78	1·81	1·85	1·89	1·92
53	1·74	1·78	1·82	1·85	1·88
54	1·71	1·74	1·79	1·81	1·84
55	1·66	1·72	1·75	1·78	1·80

TABLE 23.

Western Declination of Magnetic Needle,

For Central Europe, at the beginning of 1870.

Declination diminishes about 0°·16 yearly.

North Latitude.	Longitude East from Ferro.										
	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°	30°
45°	16·9	16·5	16·1	15·6	15·2	14·7	14·2	13·8	13·3	12·9	12·4
50	18·4	17·9	17·4	16·8	16·3	15·7	15·1	14·5	14·1	13·7	13·1
55	20·0	19·2	18·5	17·8	17·2	16·6	15·9	15·3	14·8	14·3	13·7
	30°	31°	32°	33°	34°	35°	36°	37°	38°	39°	40°
45°	12·4	11·9	11·4	10·9	10·4	10·0	9·7	9·2	8·8	8·4	7·8
50	13·1	12·6	12·0	11·5	11·0	10·5	9·9	9·4	8·9	8·4	7·8
55	13·7	13·1	12·6	12·1	11·5	11·0	10·4	9·8	9·3	8·7	8·0

TABLE 23a.

Western Declination of Magnetic Needle.

North Latitude.	Longitude West from Greenwich.										
	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	10°
50°	19·24	19·80	20·20	20·68	21·20	21·70	22·20	22·72	23·22	23·80	24·24
55	20·60	21·19	21·65	22·20	22·70	23·25	23·91	24·52	25·30	26·13	26·93
60	22·00	22·57	23·14	23·83	24·57	25·33	26·20	27·00	28·00	29·00	29·95

Horizontal Intensity of Terrestrial Magnetism for 1870.

West Longitude from Greenwich	North Latitude.										
	50°	57°	52°	53°	54°	55°	56°	57°	58°	59°	60°
0°	1·82	1·79	1·76	1·73	1·70	1·67	1·64	1·61	1·58	1·55	1·52
5	1·77	1·74	1·72	1·69	1·66	1·63	1·60	1·57	1·54	1·52	1·49
10	1·73	1·70	1·68	1·65	1·62	1·59	1·57	1·54	1·52	1·49	1·47

TABLE 24.

Galvanic Resistances compared to that of a Column of Mercury at 0° C.

An increment of temperature of 1° C. at a mean temperature, produces an increased resistance in

Pure Solid Metals	of about	0·4	per cent.
German Silver	,,	0·04	,,
Mercury	,,	0·08	,,
Sulphuric Acid mean	,,	1·0	,,

Calling the value given in the Table for any substance s, the resistance w of a column of l metres length, and q □ mm. section, expressed in Siemens's mercury-units, will be $w = \frac{ls}{q}$. For instance, the resistance of a pure copper wire of 0·5 □ mm. section and 10 metres length at 0° = $\frac{10 \times 0{\cdot}0162}{0{\cdot}5} = 0{\cdot}324$ Siem., and at 20° C. $0{\cdot}324 + 0{\cdot}324 \times 0{\cdot}004 \times 20 =$ 0·350 Siem. These numbers (according to Matthiessen) refer only to the pure metals, and must with commercial metals be regarded as mere approximations. That of commercial copper especially is frequently much greater.

Resistances of Metals.

Antimony (pressed)	0·360	Lead (pressed)	0·199
Bismuth ,,	0·133	Mercury	1·0000
Brass (hard)	0·051	Platinum (soft)	0·0918
Copper ,,	0·0166	Silver ,,	0·0153
,, (soft)	0·0162	,, (hard)	0·0166
German Silver (hard)	0·212	Tin	0·134
Gold	0·0209	Zinc	0·0571
Iron (soft)	0·0986		

TABLE 24a.

Resistances of Liquids.

Sulphuric Acid at 22° C., compared to Mercury at 0° C. (MM. Kohlrausch and Nippoldt).

Specific Gravity at 18°·5.	Percentage of H_2SO_4.	Resistance at 22° C.	Percentage Increment of Conductivity for 1° C.
0·9985	0·0	746300	0·47
1·0000	0·2	465100	0·47
1·0504	8·3	34530	0·653
1·0989	14·2	18946	0·646
1·1431	20·2	14990	0·799
1·2045	28·0	13133	1·317
(1·23	Minimum	12600 at	20°)
1·2631	35·2	13132	1·259
1·3163	41·5	14286	1·410
1·3547	46·0	15762	1·674
1·3994	50·4	17726	1·582
1·4482	55·2	20796	1·417
1·5026	60·3	25574	1·794

Solutions.	Temperature.	Resistance Hg = 1.
$ZnSO_4$ + 23 H_2O	23° C.	194400
,, 24 ,,	,,	191000
,, 105 ,,	,,	354000
$CuSO_4$ 45 ,,	22°	202410
,, 105 ,,	,,	339341
HCl 15 ,,	23°	13626
,, 500 ,,	,,	86679
Nitric Acid (com.)		16000
Copper Sulphate 20 per cent.	20°	170000
Zinc ,, 30 ,,	,,	216000

TABLE 24b.

Resistances of Metals (J. C. Maxwell).

"In the following Table R is the resistance in Ohms of a column 1 metre long, and 1 gramme weight, at 0° C.; and r is the resistance in centimetres per second of a cube of one centimetre, according to the experiments of Matthiessen."

	Specific Gravity.	R.	r	Percentage Increment of Resistance for 1° C at 20° C.
Silver	10·50 hard drawn	0·1689	1609	0·377
Copper	8·95 ,,	0·1469	1642	0·388
Gold	19·27 ,,	0·4150	2154	0·365
Lead	11·391 pressed	2·257	19847	0·387
Mercury	13·595 liquid	13·071	96146	0·072
Gold 2, Silver 1	15·218 hard or annealed	1·668	10988	0·065
Selenium at 100° C. crystalline form			6×10^{13}	1·00

TABLE 24*c*.

Electromotive Force of Constant Batteries (J. C. Maxwell).

			Concentrated Solution of		Volt.
Daniell I.	Amalgamated Zinc	$H_2SO_4 + 4$ Aq	$CuSO_4$	Copper	1·079
,, II.	,,	,, $+ 12$ Aq	$CuSO_4$	,,	0·978
,, III.	,,	,, $+$,,	$CuNO_3$	,,	1·00
Bunsen I.	,,	,, $+$,,	HNO_3	Carbon	1·964
,, II.	,,	,, $+$,, sp. gr. 1·38		,,	1·888
Grove	,,	,, $+ 4$ Aq	HNO_3	Platinum	1·956

A Volt. $= 1 \dfrac{\text{Cm.}^{\frac{3}{2}}\ \text{Mgr.}^{\frac{1}{2}}}{\text{Sec.}^2}$. (See pp. 202, 240.)

TABLE 25.

Comparison of Measures of Electric Current-Strength.

A Current-Strength which is measured in—	Must be multiplied by the following Numbers to reduce it to—				
	Cubic Cm. Water Gases per Minute.	Mgr. Water per Minute.	Mgr. Copper per Minute.	Mgr. Silver per Minute.	Magnetic Measure $\frac{\text{Mm.}^{\frac{1}{2}}\ \text{Mgr.}^{\frac{1}{2}}}{\text{Sec.}^2}$
Cub. Cm. Water Gases per min.	...	0·5363	1·889	6·432	0·9579
Mgr. Water per min. .	1·865	...	3·522	11·99	1·786
Mgr. Copper ,, .	0·5294	0·2839	...	3·405	0·5071
Mgr. Silver ,, .	0·1555	0·0834	0·2937	...	0·1489
Magnetic Measure $\frac{\text{Mm.}^{\frac{1}{2}}\ \text{Mgr.}^{\frac{1}{2}}}{\text{Sec.}^2}$ . . .	1·044	0·5599	1·972	6·714	

TABLE 25*a*.

Birmingham Wire Gauge (Holtzapffel).

BWG.	Diameter in inches.	BWG.	Diameter in inches.	BWG.	Diameter in inches.
0	0·340	14	0·083	28	0·014
2	·284	16	·065	30	·012
4	·238	18	·049	32	·009
6	·203	20	·035	33	·008
8	·165	22	·028	34	·007
10	·134	24	·022	35	·005
12	·109	26	·018	36	·004

TABLE 25*b*.

Mean Specific Heats of Water and Platinum.

Water (Regnault).		Platinum (Pouillet).	
From 0° to 40° C.	1·0013	From 0° to 100° C.	0·0335
,, 0 ,, 80	1·0035	,, 0 ,, 300	0·0343
,, 0 ,, 120	1·0067	,, 0 ,, 500	0·0352
,, 0 ,, 160	1·0109	,, 0 ,, 700	0·0360
,, 0 ,, 200	1·0160	,, 0 ,, 1000	0·0373
,, 0 ,, 230	1·0204	,, 0 ,, 1200	0·0382

TABLE 25*c*.

Heat of Combustion in Oxygen (Stewart).

Substance burned.	Grammes of Water raised 1° C. by Combination of 1 Gramme.	Compound formed.	Observer.
Hydrogen	34462	OH_2	Favre and Silbermann.
,, ...	33808	,,	Andrews.
Carbon	8080	CO_2	Favre and Silbermann.
,, ...	7900	,,	Andrews.
Sulphur	2220	SO_2	Favre and Silbermann.
,, ...	2307	,,	Andrews.
Phosphorus . . .	5747	P_2O_5	,,
Zinc	1301	ZnO	,,
Iron	1576	Fe_3O_4	,,
Tin	1233	SnO_2	,,
Copper	602	CuO	,,
Carbonic Oxide (CO)	2431	CO_2	,,
,,	2403	,,	Favre and Silbermann.
Marsh Gas (CH_4) .	13063	$CO_2 + 2OH_2$	,, ,,
,, .	13108	,,	Andrews.
Olefiant Gas (C_2H_4) .	11942	$2CO_2 + 2OH_2$	,,
,, .	11858	,,	Favre and Silbermann.
Alcohol (C_2H_5HO) .	6850	$2CO_2 + 3OH_2$	Andrews.
,, .	7183	,,	Favre and Silbermann.

Heat of Combustion in Chlorine.

Hydrogen	23783	HCl	Favre and Silbermann.
Potassium	2655	KCl	Andrews.
Zinc	1529	$ZnCl$	,,
Iron	1745	Fe_2Cl_6	,,
Tin	1079	$SnCl_4$	,,
Antimony	707	$SbCl_3$	,,
Arsenic	994	$AsCl_3$	,,
Copper	961	$CuCl_2$	,,

TABLE 26.

Symbols, Atomic Weight, Valence, and Specific Heat of some Elements.

The Atomic Weight is the smallest proportion in which the element enters into combination; hydrogen being taken as 1.

The Valence or Atomicity indicates the number of atoms of hydrogen or other univalent element which one atom will replace or combine with. In some cases the element acts as if its valence were less by 2 or 4 than that given. Equal quantities of electricity liberate equal valences.

Specific Heat multiplied by atomic weight is nearly constant for the same physical state in all elements.

The Electro-negative elements, or those which in electrolysis appear at the positive pole or zincode, are printed in italics; the electro positive, or those which appear at the negative pole or platinode, in Roman type. The difference, however, is only one of degree.

Name.	Symbol.	Atomic Weight.	Valence.	Specific Heat of Equal Parts.
Aluminium	Al	27·5	VI. III.	0·2143
Antimony	Sb	122	V.	0·0508
Arsenic	As	75	V.	0·0814
Barium	Ba	137	II.	
Bismuth	Bi	208	V.	0·0308
Boron	B	11	IV.	
Bromine	Br	80	I.	0·1060 Liquid. 0·0843 Solid.
Calcium	Ca	40	II.	
Carbon	C	12	IV.	0·2415 Charcoal. 0·1468 Diamond.
Chlorine	Cl	35·5	I.	0·1210 Const. press.
Chromium	Cr	52·5	VI.	
Cobalt	Co	58·8	VI.	0·1070
Copper	Cu	63·5	II.	0·0951
Gold	Au	196·7	III.	0·0324
Hydrogen	H	1	I.	3·4090 Const. press.
Iodine	I	127	I.	0·0541
Iron	Fe	56	VI. III.	0·1138
Lead	Pb	207	IV.	0·0314
Magnesium	Mg	24	II.	0·2499
Manganese	Mn	55	VI. III.	0·114
Mercury	Hg	200	II.	0·0319 Solid.
Nickel	Ni	58·8	VI. III.	0·1091
Nitrogen	N	14	V.	0·2438 Const. press.
Oxygen	O	16	II.	0·2175 ,,
Phosphorus	P	31	V.	
Platinum	Pt	197·4	IV.	0·0324
Potassium	K	39	I.	0·1696
Silicon	Si	28·5	IV.	
Silver	Ag	108	I.	0·0570
Sodium	Na	23	I.	0·2934
Strontium	Sr	87·5	II.	
Sulphur	S	32	VI.	0·1776
Tin	Sn	118	IV.	0·0562
Zinc	Zn	65	II.	0·0955

TABLE 27.

Numbers frequently required.

(The fractions in brackets are approximate values.)

$\pi = 3{\cdot}1416\ (\frac{22}{7}),\ \pi^2 = 9{\cdot}870,\ \frac{1}{\pi} = 0{\cdot}3183,\ \frac{\pi}{4} = {\cdot}7854,\ \log\pi = 0{\cdot}4971499.$

The modulus of natural logarithms = 2·3026.
The angle of which the arc is equal to the radius = $57^\circ{\cdot}2958$ = $3437'{\cdot}75 = 206265''$.
Ratio of the probable to the mean error = 0·67449 ($\frac{2}{3}$).

1 Paris foot	= 0·32484 metre ($\frac{13}{40}$).	1 metre	= 3·0784 Paris feet.
1 Paris line	= 2·2588 mm. ($\frac{9}{4}$).	1 mm.	= 0·44330 Paris line.
1 Rhenish foot	= 0·31385 metre ($\frac{10}{32}$).	1 metre	= 3·1862 Rhenish feet.
1 English foot	= 0·30479 ,, ($\frac{7}{23}$).	1 metre	= 3·2809 English feet.
1 Ger. mile	= 7·4204 kilom. ($\frac{30}{4}$).	1 kil.	= 0·13476 Ger. mile.
1 English mile	= 1·60929 ,,	1 kil.	= 0·62138 English mile ($\frac{5}{8}$).

Half the major axis of the earth = 6377400 metres.
,, minor ,, = 6356100 ,,
The mean semidiameter of ,, = 6366800 ,,

Accelerative force of Gravity.	Length of Seconds Pendulum.
At 45° latitude 9806 mm. =	ft. 993·5 mm.
,, the equator 9780 ,,	990·9 ,,
,, the poles 9832 ,,	996·2 ,,

Mean length of civil year 365 dy. 5 hr. 48 m. 48 sec.

Velocity of sound at 0° C. in dry air = $330\ \frac{\text{Mtr.}}{\text{Sec.}}$

The coefficient of expansion of gases 0·003665 ($\frac{1}{273}$).
Latent heat of water = 79·4.
Latent heat of steam at 100° C. = 540.
Specific heat of air at constant pressure = 0·237.
Atomic weight divided by vapour-density, compared to air, gives 28·88.
Vapour-density compared to hydrogen = molecular weight.
1 litre of hydrogen at 0° and 760 mm. weighs 0·0896 grm.
Mechanical equivalent of heat—

1 ℔ water heated 1° Fahr. = 772 foot pounds.
1 ℔ ,, 1° C. = 1390 ,,
1 grm ,, 1° C. = 424 gramme metres = $4157 \cdot \frac{\text{Mtr.}^2\ \text{Grm.}}{\text{Sec.}^2}$

A galvanic current of $1\ \frac{\text{Mm.}^{\frac{1}{2}}\ \text{Mgr.}^{\frac{1}{2}}}{\text{Sec.}^2}$ decomposes 0·560 mgr. of water per minute.

Siemens's mercury unit of resistance is in absolute measure 0·9705 Ohm or B A unit.

$1\ \text{Ohm} = 1\ \frac{\text{Earth quadrant}}{\text{Second}} = 10^7\ \frac{\text{Metre}}{\text{Seconds}}.$

1 Volt. = 100000 absolute electromagnetic units of tension, or $1\ \frac{\text{Cm.}^{\frac{3}{2}}\ \text{Mgr.}^{\frac{1}{2}}}{\text{Sec.}^2}$.

Wave-length of sodium light (D. Fraunhofer) = 0·0005895 mm.

A plate of quartz mm. thick rotates the plane of polarisation of sodium light 21°·67.

TABLE 28.

SQUARES, SQUARE ROOTS, AND RECIPROCALS.

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	n	n^2	$\sqrt{n}$	$\frac{1}{n}$
1	1	1·000	1·0000	50	2500	7·071	0·0200
2	4	1·414	0·5000	51	2601	7·141	0·0196
3	9	1·732	0·3333	52	2704	7·211	0·0192
4	16	2·000	0·2500	53	2809	7·280	0·0189
5	25	2·236	0·2000	54	2916	7·348	0·0185
6	36	2·449	0·1667	55	3025	7·416	0·0182
7	49	2·646	0·1429	56	3136	7·483	0·0179
8	64	2·828	0·1250	57	3249	7·550	0·0175
9	81	3·000	0·1111	58	3364	7·616	0·0172
10	100	3·162	0·1000	59	3481	7·681	0·0169
11	121	3·317	0·0909	60	3600	7·746	0·0167
12	144	3·464	0·0833	61	3721	7·810	0·0164
13	169	3·606	0·0769	62	3844	7·874	0·0161
14	196	3·742	0·0714	63	3969	7·937	0·0159
15	225	3·873	0·0667	64	4096	8·000	0·0156
16	256	4·000	0·0625	65	4225	8·062	0·0154
17	289	4·123	0·0588	66	4356	8·124	0·0152
18	324	4·243	0·0556	67	4489	8·185	0·0149
19	361	4·359	0·0526	68	4624	8·246	0·0147
20	400	4·472	0·0500	69	4761	8·307	0·0145
21	441	4·583	0·0476	70	4900	8·367	0·0143
22	484	4·690	0·0455	71	5041	8·426	0·0141
23	529	4·796	0·0435	72	5184	8·485	0·0139
24	576	4·899	0·0417	73	5329	8·544	0·0137
25	625	5·000	0·0400	74	5476	8·602	0·0135
26	676	5·099	0·0385	75	5625	8·660	0·0133
27	729	5·196	0·0370	76	5776	8·718	0·0132
28	784	5·292	0·0357	77	5929	8·775	0·0130
29	841	5·385	0·0345	78	6084	8·832	0·0128
30	900	5·477	0·0333	79	6241	8·888	0·0127
31	961	5·568	0·0323	80	6400	8·944	0·0125
32	1024	5·657	0·0313	81	6561	9·000	0·0123
33	1089	5·745	0·0303	82	6724	9·055	0·0122
34	1156	5·831	0·0294	83	6889	9·110	0·0120
35	1225	5·916	0·0286	84	7056	9·165	0·0119
36	1296	6·000	0·0278	85	7225	9·220	0·0118
37	1369	6·083	0·0270	86	7396	9·274	0·0116
38	1444	6·164	0·0263	87	7569	9·327	0·0115
39	1521	6·245	0·0256	88	7744	9·381	0·0114
40	1600	6·325	0·0250	89	7921	9·434	0·0112
41	1681	6·403	0·0244	90	8100	9·487	0·0111
42	1764	6·481	0·0238	91	8281	9·539	0·0110
43	1849	6·557	0·0233	92	8464	9·592	0·0109
44	1936	6·633	0·0227	93	8649	9·644	0·0108
45	2025	6·708	0·0222	94	8836	9·695	0·0106
46	2116	6·782	0·0217	95	9025	9·747	0·0105
47	2209	6·856	0·0213	96	9216	9·798	0·0104
48	2304	6·928	0·0208	97	9409	9·849	0·0103
49	2401	7·000	0·0204	98	9604	9·899	0·0102
50	2500	7·071	0·0200	99	9801	9·950	0·0101
				100	10000	10·000	0·0100

TABLE 29.—Trigonometrical Functions.

Angle.	Sine.		Tangent.		Cotangent.		Cosine.		
0°	0·000		0·000		∞		1·000		90°
1	0·017	17	0·017	17	57·29		1·000	0	89
2	0·035	18	0·035	18	28·64		0·999	0	88
3	0·052	17	0·052	17	19·08		0·999	0	87
4	0·070	18	0·070	18	14·30		0·998	1	86
5	0·087	17	0·087	17	11·43		0·996	2	85
6	0·105	18	0·105	18	9·514		0·995	1	84
7	0·122	17	0·123	18	8·144		0·993	2	83
8	0·139	17	0·141	18	7·115		0·990	3	82
9	0·156	17	0·158	17	6·314	811	0·988	2	81
10°	0·174	18	0·176	18	5·671	643	0·985	3	80°
11	0·191	17	0·194	18	5·145	526	0·982	3	79
12	0·208	17	0·213	19	4·705	440	0·978	4	78
13	0·225	17	0·231	18	4·331	374	0·974	4	77
14	0·242	17	0·249	18	4·011	320	0·970	4	76
15	0·259	17	0·268	16	3·732	279	0·966	4	75
16	0·276	17	0·287	19	3·487	245	0·961	5	74
17	0·292	16	0·306	19	3·271	216	0·956	5	73
18	0·309	17	0·325	19	3·078	193	0·951	5	72
19	0·326	17	0·344	19	2·904	174	0·946	5	71
20°	0·342	16	0·364	20	2·747	157	0·940	6	70°
21	0·358	16	0·384	20	2·605	142	0·934	6	69
22	0·375	17	0·404	20	2·475	130	0·927	7	68
23	0·391	16	0·424	20	2·356	119	0·921	6	67
24	0·407	16	0·445	21	2·246	110	0·914	7	66
25	0·423	16	0·466	21	2·145	101	0·906	8	65
26	0·438	15	0·488	22	2·050	95	0·899	7	64
27	0·454	16	0·510	22	1·963	87	0·891	8	63
28	0·469	15	0·532	22	1·881	82	0·883	8	62
29	0·485	16	0·554	22	1·804	77	0·875	8	61
30°	0·500	15	0·577	23	1·732	73	0·866	9	60°
31	0·515	15	0·601	24	1·664	67	0·857	9	59
32	0·530	15	0·625	24	1·600	64	0·848	9	58
33	0·545	15	0·649	24	1·540	60	0·839	9	57
34	0·559	14	0·675	26	1·483	57	0·829	10	56
35	0·574	15	0·700	25	1·428	55	0·819	10	55
36	0·588	14	0·727	27	1·376	52	0·809	10	54
37	0·602	14	0·754	27	1·327	49	0·799	10	52
38	0·616	14	0·781	27	1·280	47	0·788	11	52
39	0·629	13	0·810	29	1·235	45	0·777	11	51
40°	0·643	14	0·839	29	1·192	43	0·766	11	50°
41	0·656	13	0·869	30	1·150	42	0·755	11	49
42	0·669	13	0·900	31	1·111	39	0·743	12	48
43	0·682	13	0·933	33	1·072	39	0·731	12	47
44	0·695	13	0·966	32	1·036	36	0·719	12	46
45°	0·707	12	1·000	35	1·000	36	0·707	12	45°
	Cosine.		Cotangent.		Tangent.		Sine.		Angle

TABLE 30.

Logarithms to 4 Places.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	42
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	23
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4988	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5337	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
N.	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE 30—*Continued.*

LOGARITHMS TO 4 PLACES.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
88	9445	9450	9555	9460	9465	9469	9474	9479	9484	9489	5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	5
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4
N.	0	1	2	3	4	5	6	7	8	9	Diff.

INDEX.

THE END.

100 rel
nel
worth
2 50 new